Multidimensional Signal Processing

R.J. Marks II Lecture Notes

Dudgeon & Mersereau

University of Washington (1984)

ENGINEERING LIBRARY RESERVE FORM

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Quarter Autumn '86
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Phone Number 543-6990
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Number of students in class 15 (?)

Leave space blank	CALL #	AUTHOR	TITLE
		D.E. Dudgeon & R.M. Mersereau	Multidimensional Digital Signal Processing (Prentice Hall, 1984) ISBN #0-13-604959-1
		R.N. Bracewell	The Fourier Transform and Its Applications, 2nd Edition (McGraw Hill, 1978) ISBN #0-07-007013-X
		N.K. Bose, Ed	Multidimensional Systems: Theory and Practice (IEEE Press)
		H. Lee, Ed	Imaging Technology (IEEE Press)
		T.S. Huang, Ed.	Image Reconstruction From Incomplete Observations JAI Press
		T.S. Huang, Ed.	Image Enhancement and Restoration JAI Press

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EE595

Text: Dudgeon & Mesereau
2nd: Papoulis (Helpful, but not required)
reserve list →

Grading:
0 H.W: work together : 10% (Graded spotwise)
30 Midterm : ~~15%~~ 30
30 Oral Report (Final 2 weeks) : 20%
40 Final : 40%

Homework

1. Chapt 1 : 1, 2, 3, 7, 8

2. Chapt 1 : 12*, 16, 14*†, 15b.c, 17, 20, 22

* do in M dimensions
† Bracewell shows that

$$\int_{\vec{x}} f(r)\, e^{\pm j 2\pi \vec{u}^T \vec{x}}\, d\vec{x} = \frac{2\pi}{q^{\frac{m}{2}-1}} \int_0^\infty f(r)\, J_{\frac{m}{2}-1}(2\pi q r)\, r^{\frac{m}{2}}\, dr$$

$$r = \sqrt{\sum_{n=1}^{m} x_m} = \|\vec{x}\| \quad ; \quad q = \|\vec{u}\|$$

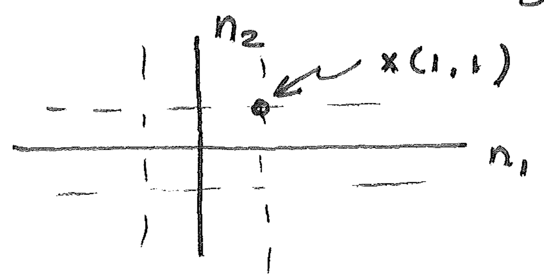
I. Multidimensional Signals & Systems

1.1. 2-D discrete signals

$$x[n_1, n_2] \; ; \quad -\infty < n_1, n_2 < \infty$$

Each ~~co~~ integer # pair has a # (possibly complex) assigned to it.

Can visualize in a rectangular grid:



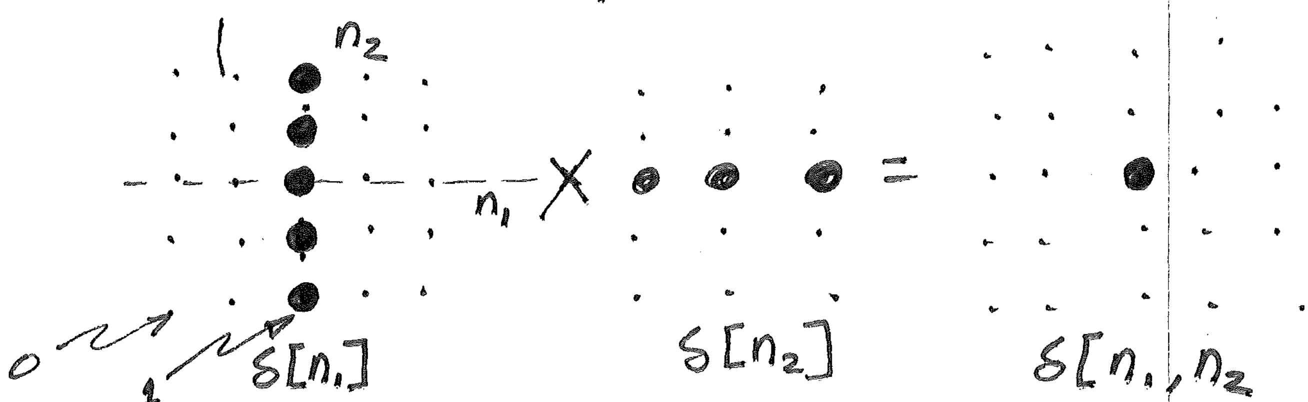
But, not nece. so (e.g. hexogonal grid).

1.1.1. Some Special Sequences

(a) Unit impulse (2-D Kronecker delta)

$$\delta[n_1, n_2] = \begin{cases} 1 & ; n_1 = n_2 = 0 \\ 0 & ; \text{otherwise} \end{cases}$$

$$= \delta[n_1]\,\delta[n_2]$$



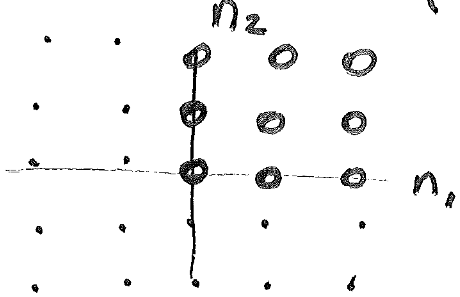
Sifting Property: $\sum_n u[n]\,\delta[m-n] = u[m]$

$$\sum_{n_1} \sum_{n_2} u[n_1, n_2]\,\delta[m_1 - n_1, m_2 - n_2] = u[m_1, m_2]$$

(elaborate)

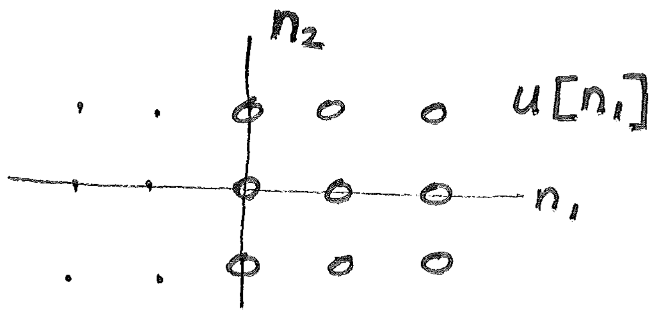
(b) 2-D unit step

$$u[n_1, n_2] = \begin{cases} 1 & ; \; n_1 \geq 0 \text{ and } n_2 \geq 0 \\ 0 & ; \text{ otherwise} \end{cases}$$



$$u[n_1, n_2] = u[n_1]\, u[n_2]$$

$$u[n] = \begin{cases} 1 & ; \; n \geq 0 \\ 0 & ; \; n > 0 \end{cases}$$



(c) Exponential Sequences

$$x[n_1, n_2] = a^{n_1} b^{n_2} \qquad -\infty < n_1, n_2 < \infty$$

a & b can be complex

Special case:

$$a = e^{j\omega_1}, \quad b = e^{j\omega_2}$$

then $x[n_1, n_2] = e^{j(\omega_1 n_1 + \omega_2 n_2)}$

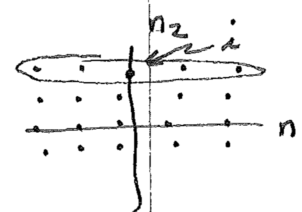
1.1.2. Separable Sequences

Separable if

$$x[n_1, n_2] = x_1[n_1]\, x_2[n_2]$$

<u>Any</u> 2-D sequence can be written as a sum of seperable sequences:

$$x[n_1, n_2] = \sum_{i=-\infty}^{\infty} x_{i1}[n_1]\, x_{i2}[n_2]$$



e.g. choose
$$x_{i1}[n_1] = x[n_1, i]$$
$$x_{i2}[n_2] = \delta[n_2 - i]$$

Then:

$$\sum_{i=-\infty}^{\infty} x_{i1}[n_1]\, x_{i2}[n_2] = \sum_{i=-\infty}^{\infty} x[n_1, i]\, \delta[n_2 - i]$$

$$= \cancel{\Sigma} \quad = x[n_1, n_2]$$

(by shifting property)

If $x[n_1 n_2]$ has only a finite # of nonzero elements, then the i sum can be over a finite range (elaborate)

1.1.3. FINITE-EXTENT SEQUENCES

$\exists$ a finite region of support outside of which $x[n_1, n_2] = 0$

ie, $\exists\ N_1\ M_1\ N_2\ M_2\ \ni$

$$x[n_1, n_2] = x[n_1, n_2]\,\mu[n_1 - N_1]\,\mu[M_1 - n_1] \times \mu[n_2 - N_2]\,\mu[M_2 - n_2]$$

(elaborate)

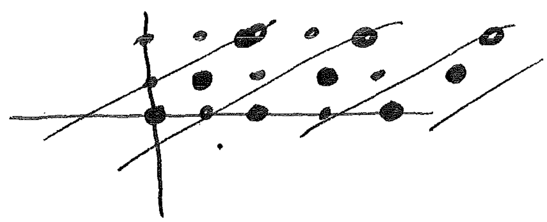
1.1.4. PERIODIC SEQUENCES

~~Doubly~~ Rectangularly Periodic:

$$\tilde{x}[n_1, n_2 + N_2] = \tilde{x}[n_1, n_2]$$

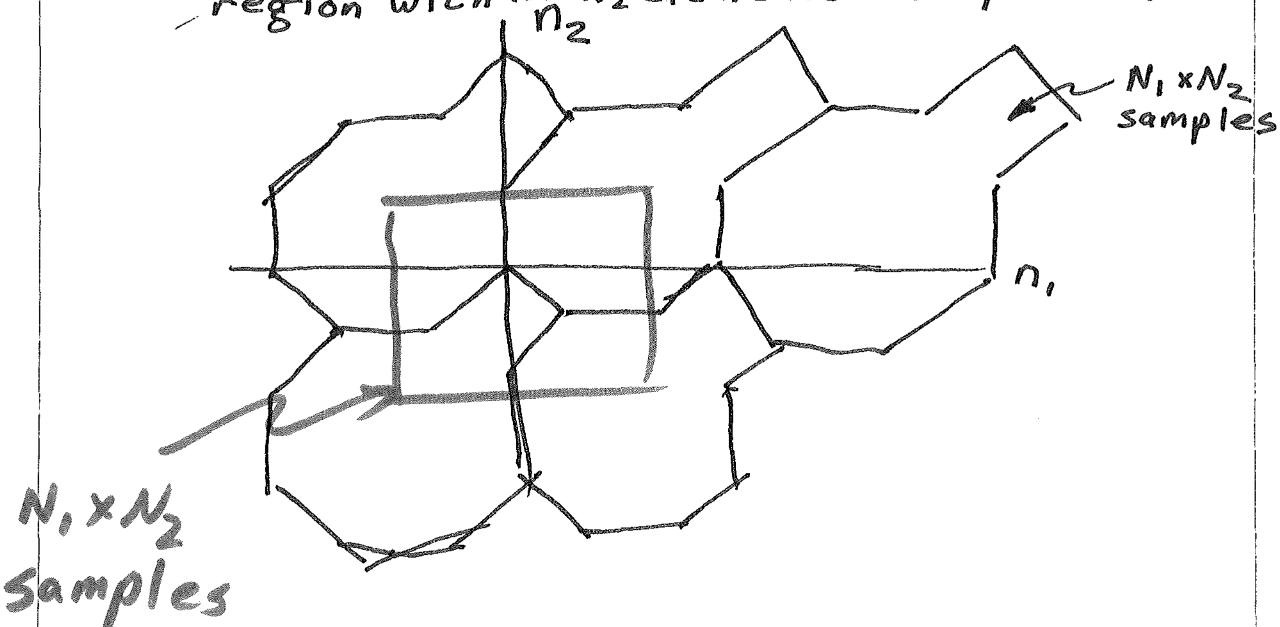
$$\tilde{x}[n_1 + N_1, n_2] = \tilde{x}[n_1, n_2]$$

N_1 = horizontal period
N_2 = vertical period



~~(elaborate)~~

~~Note: periodicity is in $N_1 \times N_2$ blocks. (rectangles).~~ This is ~~not~~, however, not the only possibility. ~~Ex.~~ Any connected region with $N_1 \times N_2$ elements is a period:



More general definition:

$$\tilde{x}(n_1 + N_{11}, n_2 + N_{21}) = \tilde{x}(n_1, n_2)$$
$$\tilde{x}(n_1 + N_{12}, n_2 + N_{22}) = \tilde{x}(n_1, n_2)$$

Condition: $D = N_{11} N_{22} - N_{12} N_{21} \neq 0$

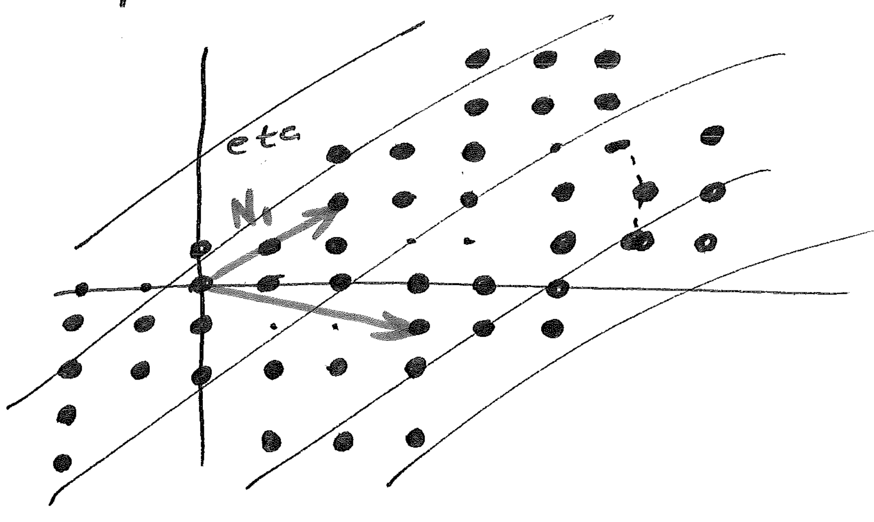
Note:

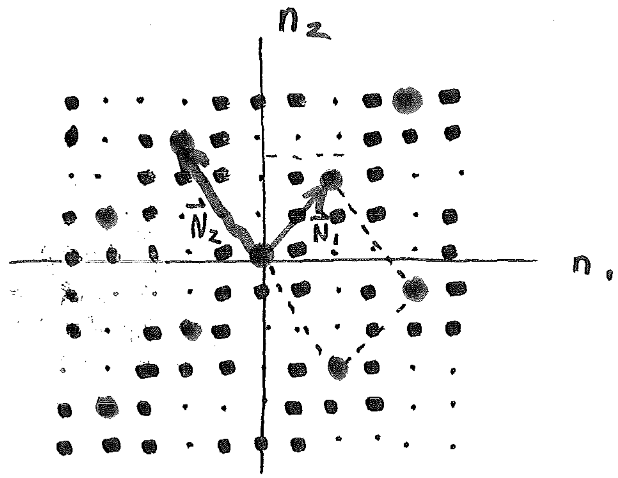
$$D = \det \underline{\mathbf{N}} = \det \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}$$

$$\underline{\mathbf{N}} = [\vec{N}_1 \mid \vec{N}_2]$$

$$\vec{N}_1 = \begin{bmatrix} N_{11} \\ N_{21} \end{bmatrix} \qquad \vec{N}_2 = \begin{bmatrix} N_{21} \\ N_{22} \end{bmatrix}$$

Interpretation:





• = ORIGIN REPEATED

$\underline{N}$ not unique (just as period isn't)

Choose closest

$\vec{N}_1 = (2,3)'$

$\vec{N}_2 = (-2,3)'$

Period is in parallelogram defined by $\vec{N}_1$ & N_2
(Connect red dots)

# of elements in period = $|D|$

Generalization to M-D case:

M-D periodic sequence:

$$\tilde{x}[\vec{n} + \vec{N}_i] = \tilde{x}[\vec{n}]$$

$\vec{n}$ = M-D coordinate vector

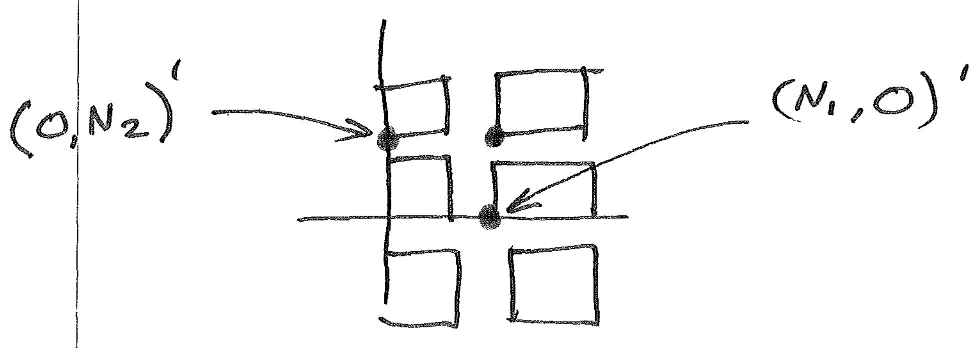
$\vec{N}_i$ = periodicity vectors

$$\underline{N} = [\vec{N}_1 \,\vdots\, \vec{N}_2 \,\vdots\, \cdots \,\vdots\, \vec{N}_M]$$

$$|D| = \underbrace{\#\text{ elements}}_{?} = \det \underline{N} \neq 0$$

Q: When does sequence become rectangularly periodic

A: When $\underline{N}$ is diagonal



(convention sloppy. Could have, e.g.

$$\begin{bmatrix} 0 & N_1 \\ N_2 & 0 \end{bmatrix}$$

Thus:

$$\begin{bmatrix} 0 & 0 & N_1 \\ N_3 & 0 & 0 \\ 0 & N_2 & 0 \end{bmatrix}$$ is also rectangularly periodic in 3-D.

Larger Periods

$\vec{r}$ = vector of integers

$$\tilde{x}[\vec{n} + \underline{N}\vec{r}] = \tilde{x}[\vec{n}]$$

Proof:

$$\tilde{x}[\vec{n} + \vec{N}_i] = \tilde{x}[\vec{n}]$$

$$\tilde{x}[\tilde{n} + r_i \vec{N}_i] = \tilde{x}[\vec{n}] \quad ; r_i = \text{integer}$$

$$\tilde{x}[\tilde{n} + \sum_i r_i \vec{N}_i] = \tilde{x}[\vec{n}]$$

$$= \tilde{x}[\vec{n} + \underline{N}\,\vec{r}]$$

Also, if $\underline{N}$ = periodicity matrix for $\tilde{x}[\vec{n}]$,
then $\underline{N}\,\underline{P}$ = " " " "
where $\underline{P}$ = matrix of integers

~~Wh.~~
~~Proof:~~

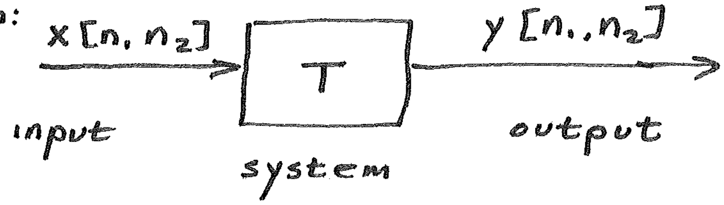
$$\cancel{\hat{\underline{N}} = [\, r_1 \vec{N}_1 \,\vdots\, r_2 \vec{N}_2 \,\vdots\, r_3 \vec{N}_3 \,\vdots\, \cdots \,\vdots\, r_M \vec{N}_M \,]}$$

$$\cancel{= \vec{N} \begin{bmatrix} r_1 & & 0 \\ & r_2 & \\ 0 & & \ddots \; r_M \end{bmatrix}}$$

$$\cancel{\hat{\underline{N}} =}$$

1.2. MULTIDIMENSIONAL SYSTEMS

2-D system:



$$y = Tx \quad \text{(in } M \text{ dimensions)}$$

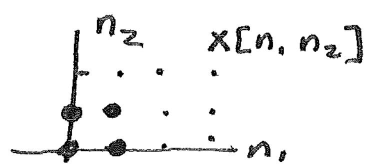
1.2.1. Fundemental Operations on Multidimensional Systems

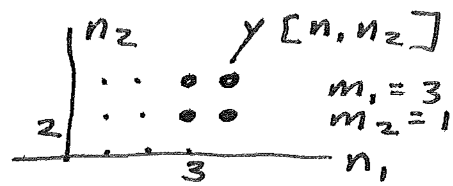
(a) Addition $\quad y[n_1, n_2] = x[n_1, n_2] + w[n_1, n_2]$

(b) Multipication $\quad y[n_1, n_2] = c\, x[n_1, n_2]$

spatially varying gain $\quad y[n_1, n_2] = x[n_1, n_2]\, w[n_1, n_2]$

(c) Shifting $\quad y[n_1, n_2] = x[n_1 - m_1, n_2 - m_2]$





(d) Memoryless nonlinearity

$$y = Tx$$

$y[n_1, n_2]$ depends <u>only</u> on $x[n_1, n_2]$

(Also called ZNL)

<u>Ex</u> $\quad y[n_1, n_2] = f\, x[n_1, n_2]$

function

$$y[n_1, n_2] = x^2[n_1, n_2]$$

(e) Sifting property of impulse:

$$x[n_1, n_2] = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} x[k_1, k_2]\, \delta[n_1 - k_1, n_2 - k_2]$$

1.2.2. LINEAR SYSTEMS

Linear $\Rightarrow$ Two Conditions $\quad y = Lx$

Homogeneity:

$$L[x_1 + x_2] = y_1 + y_2$$

Additivity:

$$Lax = aLx$$

Superposition Sum:

$$y[n_1 n_2] = L\, x_1[n_1 n_2]$$

$$= L \sum_{k_1} \sum_{k_2} x[k_1 k_2]\, \delta[n_1 - k_1, n_2 - k_2]$$

Additivity $\Rightarrow$ $$= \sum_{k_1} \sum_{k_2} L\, x[k_1 k_2]\, \delta[n_1 - k_1, n_2 - k_2]$$

Homogeneity $\Rightarrow$ $$= \sum_{k_1} \sum_{k_2} x[k_1 k_2]\, L\, \delta[n_1 - k_1, n_2 - k_2]$$

$$h_{k_1 k_2}[n_1, n_2] \equiv L\, \delta[n_1 - k_1, n_2 - k_2]$$

= (Space Variant) Impulse Response

1.2.3. Shift Invariant Systems

$$y[n_1, n_2] = T\, x[n_1\, n_2]$$

Shift-invariant if

$$T\, x[n_1 - m_1, n_2 - m_2] = y[n_1 - m_1, n_2 - m_2]$$

ie, shifting input shifts output (elaborate)

Shift invariance does not imply linearity ~~etc.~~ & visa versa

<u>Ex</u> $L\, x[n_1 n_2] = c[n_1 n_2]\, x[n_1 n_2]$

Linear?

$$L\, a x = a L x$$

$$L\, x_1 + x_2 = c[x_1 + x_2] = c x_1 + c x_2 = L x_1 + L x_2$$

Yes!

Shift-invariant?

$$L\, x[n_1, n_2] = y[n_1, n_2] = c[n_1 . n_2]\, x[n_1\, n_2]$$

$$y[n_1 - m_1, n_2 - m_2] = c[n_1 - m_1, n_2 - m_2]\, x[n_1 - m_1\, n_2 - m_2]$$

$$\neq L\, x[n_1 - m_1, n_2 - m_2] = c[n_1 n_2]\, x[n_1 - m_1\, n_2 - m_2]$$

$\therefore$ Not S.I.

<u>Ex</u> $y[n_1 n_2] = T\, x[n_1 n_2] = x^2[n_1 n_2]$

Linear?

$$T a x = a^2 x^2 \neq a y = a x^2 \quad \text{No!}$$

Shift-invariant?

$$y[n_1 - m_1, n_2 - m_2] = x^2[n_1 - m_1, n_2 - m_2]$$

$$= T\, x[n_1 - m_1, n_2 - m_2] = x^2[n_1 - m_1, n_2 - m_2]$$

Yes!

Ex: $y[n_1 n_2] = x[n_1 - N_1, n_2 - N_2] = L\, x_1[n_1, n_2]$

Linear?

$L\, ax = ay$

$L\, x_1 + x_2 = L x_1 + L x_2$ Yes!

Shift-invariant?

$L\, x[n_1 - m_1, n_2 - m_2] = x[(n_1 - m_1) - N_1, (n_2 - m_2) - N_2]$

$y[n_1 - m_1, n_2 - m_2] = x[n_1 - m_1 - N_1, n_2 - m_2 - N_2]$

They are equal $\Longrightarrow$ Yes!

1.2.4. Linear Shift-Invariant Systems

Linear:

$$y[n_1 n_2] = \sum_{k_1} \sum_{k_2} x[k_1 k_2]\, h_{k_1 k_2}[n_1 n_2]$$

$$h_{k_1 k_2}[n_1 n_2] = L\, \delta[n_1 - k_1,\ n_2 - k_2]$$

Shift invariant

$$L\, x[n_1 - k_1,\ n_2 - k_2] = y[n_1 - k_1,\ n_2 - k_2]$$

Hence:

$$L\, \delta[n_1 - k_1, n_2 - k_2] = h[n_1 - k_1, n_2 - k_2]$$

$$\cancel{= h_{00}[n_1 - k_1, n_2 - k_2]}$$

$$\cancel{= h}$$

Note:

$$h[n_1, n_2] = L\, \delta[n_1, n_2] = h_{00}[n_1, n_2]$$

Thus:

$$h[n_1 - k_1, n_2 - k_2] = h_{00}[n_1 - k_1, n_2 - k_2]$$

Superposition Sum becomes Convolution Sum:

$$y[n_1 n_2] = \sum_{k_1} \sum_{k_2} x[k_1 k_2]\, h[n_1 - k_1, n_2 - k_2]$$

$$= x ** h$$

Commutative:

$$x ** h = h ** x$$

In M-dimensions:

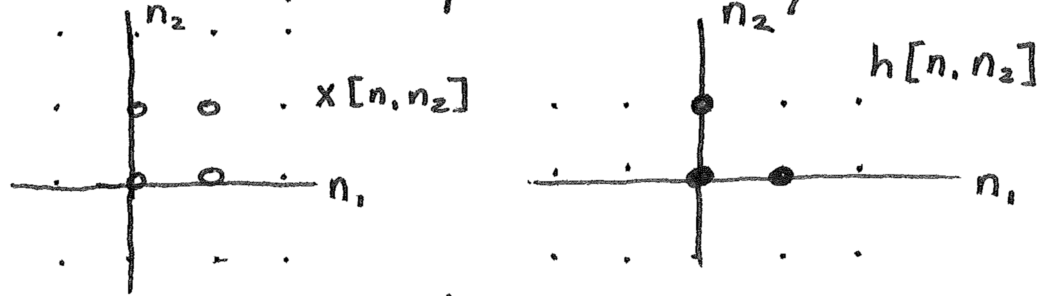
$$y[\vec{n}] = \sum_{\vec{k}} x[\vec{k}]\, h[\vec{n} - \vec{k}]$$

2-D Convolution Mechanics

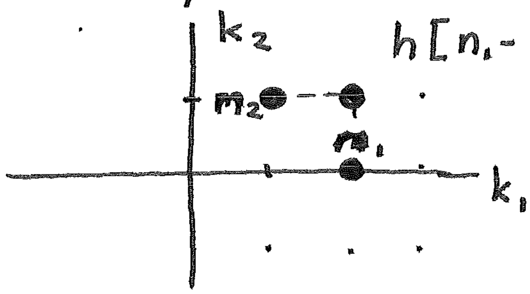
$$y[n_1,n_2] = \sum_{k_1}\sum_{k_2} x[k_1,k_2]\, h[n_1-k_1,\, n_2-k_2]$$

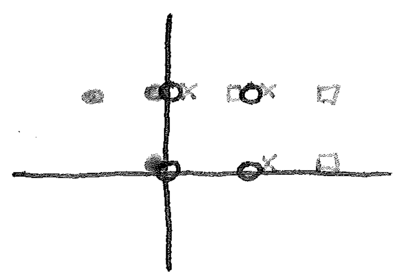
Sum is over k_1 & k_2 paramerized by n_1 & n_2

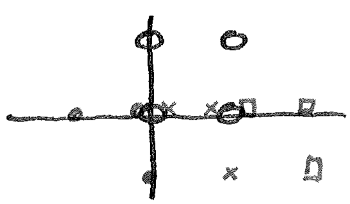
Ex

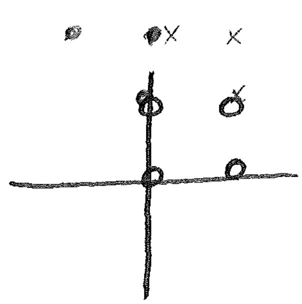


$y = x ** h$



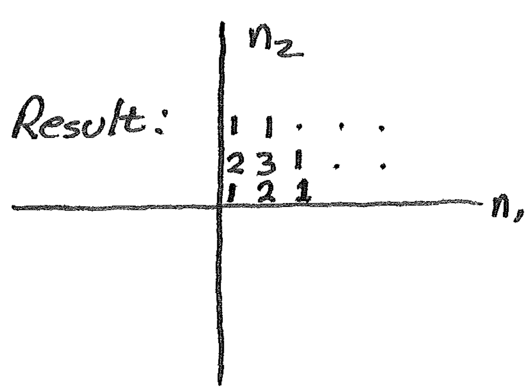






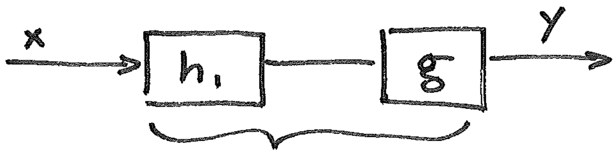
	n_1	n_2	$y[n_1,n_2]$
•	0	1	2
×	1	1	3
□	2	1	1
•	0	0	1
×	1	0	2
□	2	0	1
	0	2	1
	1	2	1

Result:



1.2.5. Cascade & Parallel Connections of Systems

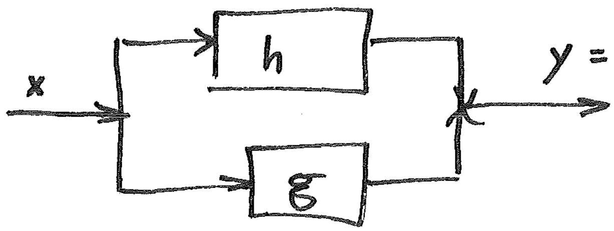
Cascade:



composite impulse response

$h_{eqiv} = h ** g$

Parallel



$y = x ** [h+g]$

$h_{equiv} = h + g$

More complicated connections (feedback)

...use Mason's GAin Rule

Separable Functions

If $x[n_1, n_2] = x_1[n_1]\, x_2[n_2]$

and $h[n_1, n_2] = h_1[n_1]\, h_2[n_2]$

then $y[n_1, n_2] = y_1[n_1]\, y_2[n_2]$

where $y_1[n_1] = x_1[n_1] * h_1[n_1]$

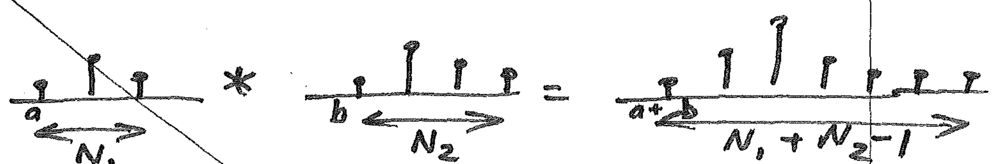
$$= \sum_{k_1=-\infty}^{\infty} x_1[k_1]\, h_1[n_1 - k_1]$$

(Omit)

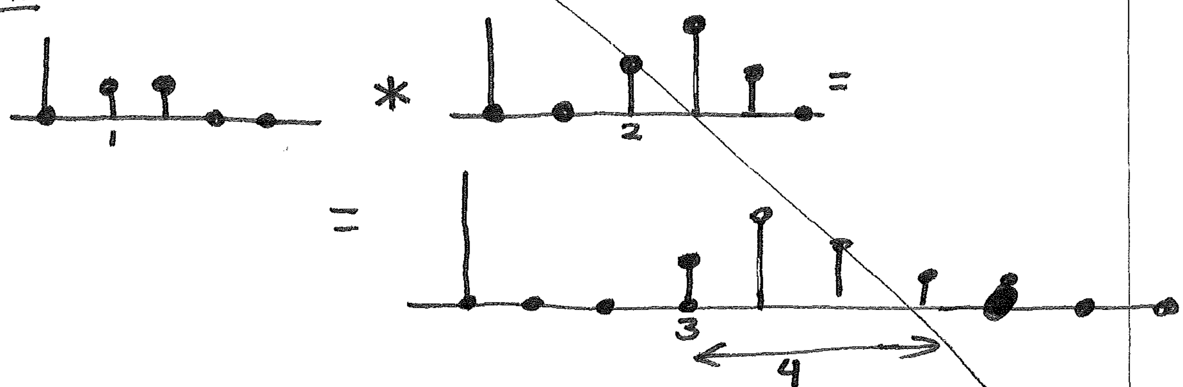
Regions of Support

Two functions with finite support will yield a third with " "

1-D case:



Ex



1.2.6. SEPARABLE SYSTEMS

$$h[n_1, n_2] = h_1[n_1]\, h_2[n_2]$$

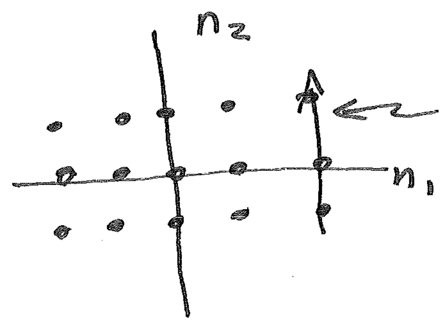
Then

$$y[n_1 n_2] = \sum_{k_1=-\infty}^{\infty} h_1(k_1) \sum_{k_2=-\infty}^{\infty} h_2[k_2]\, x[n_1-k_1, n_2-k_2]$$

Can use 2. 1-D convolutions

$$g[n_1 n_2] = \sum_{k_2=-\infty}^{\infty} h_2[k_2]\, x[n_1, n_2-k_2]$$

$$= h[n_2] \underset{n_2}{*} x[n_1, n_2]$$



fixed n_1

Convolve each column with ~~by~~ $h[n_2]$

Then

$$y[n_1 n_2] = \sum_{k_1=-\infty}^{\infty} h_1[k_1]\, g[n_1-k_1, n_2 \cancel{k_2}]$$

Convolve each <u>row</u> of $g[n_1, n_2]$ with $h_1[k_1]$

Extension to M-D

$$h[\vec{n}] = \prod_{m=1}^{M} h_m(n_m) \Leftarrow \text{seperable}$$

To convolve with $x[\vec{n}]$,

Do (a) $x[\vec{n}] * h_1[n_1]$

(b) $(\quad\quad) * h_2(n_2)$

(c) $(\quad\quad) * h_3(n_3)$

⋮

$\Rightarrow$ ~~(a)~~ HOW MANY CONVOLUTIONS
ASSUME SUPPORT WITHIN $\underbrace{N \times N \times \ldots \times N}_{M}$ cube.

(a) N^{M-1}

(b) N^{M-1}

⋮

$\Rightarrow MN^{M-1}$ convolutions needed

$\times(2N-1)$ multiplies per ~~for~~ convolutions

$= \cancel{2NM} \cdot \cancel{2N^{2-1}}$

$= 2MN^M$ multiplies

Without seperable sequence:

$$y[\vec{n}] = \sum_{k_1} \sum_{k_2} \ldots \sum_{k_m} x[n_1 \ldots n_M]\, h[\vec{n} - \vec{k}]$$

$= M$

(STABILITY)

FREQUENCY DOMAIN CHARACTERIZATION

2-D LSI system $h[n_1 n_2]$

input $x[n_1 n_2] = e^{j(\omega_1 n_1 + \omega_2 n_2)}$

$$y[n_1 n_2] = \sum_{k_1} \sum_{k_2} e^{j\omega_1[n_1 - k_1] + j\omega_2[n_2 - k_2]} x[k_1 k_2]$$

$$= e^{j(\omega_1 n_1 + \omega_2 n_2)} H(\omega_1 \omega_2)$$

$$H(\omega_1 \omega_2) = \sum_{k_1} \sum_{k_2} x[k_1 k_2] e^{-j\omega_1 n_1 - j\omega_2 n_2}$$

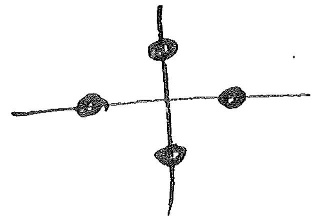
= frequency response

(2-D Fourier series)

Rectangular periodiodic.

X ÷ y periods = 2π

Ex

$h[n_1, n_2] =$ 

$$= \delta[n_1+1, n_2] + \delta[n_1-1, n_2]$$
$$+ \delta[n_1, n_2+1] + \delta[n_1, n_2-1)$$

$$H(\omega_1, \omega_2) = 2(\cos\omega_1 + \cos\omega_2)$$

separable?

Plot on p. 27

Ex $h[n_1,n_2] = a^{n_1+n_2}\, u[n_1,n_2]$

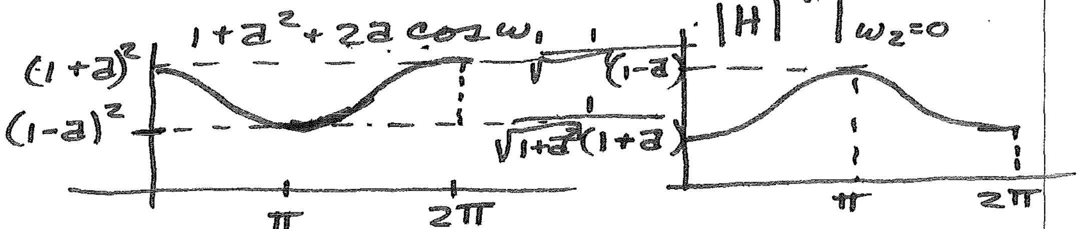
$$H(\omega_1,\omega_2) = \sum_{n_1=0}^{\infty}(a e^{j\omega_1})^{n_1} \sum_{n_2=0}^{\infty}(a e^{j\omega_2})^{n_2}$$

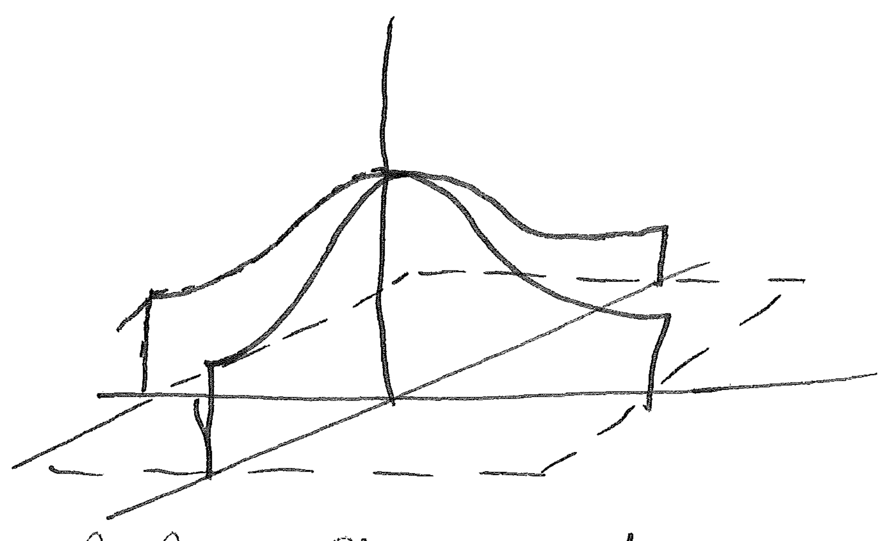
$$= \frac{1}{\cancel{a}(1+a e^{j\omega_1})(1+a e^{j\omega_2})}$$

a real

$$H(\omega_1,\omega_2) = \frac{[1+a e^{-j\omega_1}][1+a e^{j\omega_2}]}{[1+a^2+2a\cos\omega_1][1+a^2+2a\cos\omega_2]}$$

$$|H|^2 = \frac{\cancel{1+a^2}\ \ 1}{[1+a^2+2a\cos\omega_1][1+a^2+2a\cos\omega_2]}$$





For separable ⇒ always rectangular ~~structure~~ zero locus plot Yes!

Not always rect. structure (eg. Gaussian)

Generalization:

$$H(\vec{\omega}) = \sum_n h[\vec{n}]e^{-j\vec{\omega}^T\vec{n}}$$

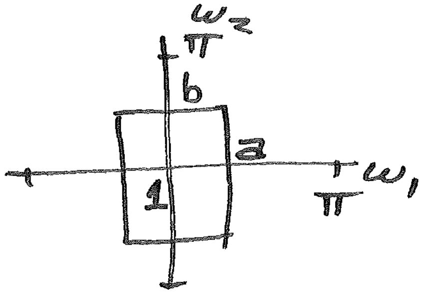
1.3.2. Finding h from H.

H is a Fourier series. $h[n_1 n_2]$'s are coefficients:

$$h[n_1,n_2] = \underbrace{\frac{1}{(2\pi)^2}}_{\text{Period}} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} H(\omega_1,\omega_2) \times e^{j(\omega_1 n_1+\omega_2 n_2)}\, d\omega_1 d\omega_2$$

Elaborate on seperable $\Rightarrow$ 2 1-D integrals

EX Ideal LPF



$$h[n_1,n_2] = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} H(\omega_1,\omega_2)e^{j(\omega_1 n_1+\omega_2 n_2)}\, d\omega_1 d\omega_2$$

seperable!

$$= \frac{1}{2\pi}\int_{-a}^{a} e^{j(\omega_1 n_1 \cancel{+\omega_2 n_2})}\, d\omega_1$$

$$* \frac{1}{2\pi}\int_{-b}^{b} e^{j\omega_2 n_2}\, d\omega_2$$

$$= \frac{\sin a n_1}{\pi n_1}\ \frac{\sin a n_2}{\pi n_2} \Leftarrow$$ What if $n_1=0$ or $n_2=0$ or both?

Bessel identities used:

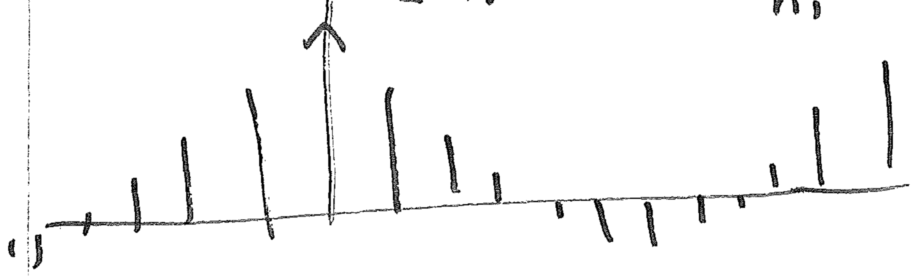
$$\int_{\phi=0}^{2\pi} e^{ja\cos\phi}\,d\phi = 2\pi J_0(a)$$

$$\int z J_0(z)\,dz = z J_1(z)$$

Thus:

$$h[n_1,n_2] = \frac{R}{2\pi}\,\frac{J_1\left(R\sqrt{n_1^2+n_2^2}\right)}{\sqrt{n_1^2+n_2^2}}$$

$$h[n_1,0] = \frac{R\,J_1[Rn_1]}{n_1}$$



Lot like sinc, <u>but</u> axis crossings aren't evenly spaced.

Bracewell defines: "jinc"

Gaskill defines "sombrero"

Assignment: Read pp. 33-35 on properties of 2-D transform:

$$X(\omega_1,\omega_2)=\cancel{\frac{1}{(2\pi)^2}\oint}\sum_{n_1}\sum_{n_2}x[n_1,n_2]\,e^{j(\omega_1 n_1+\omega_2 n_2)}$$

In M-D:

$$X(\vec{\omega})=\sum_{n_1}\sum_{n_2}\cdots\sum_{n_m}x[\vec{n}]\,e^{j\vec{\omega}^T\vec{n}}$$

$$x[\vec{n}]=\frac{1}{(2\pi)^M}\sum_{n_1}\cdots\sum_{n_m}X(\vec{\omega})\,e^{-j\vec{\omega}^T\vec{n}}$$

1.4. SAMPLING CONTINUOUS 2-D SIGNALS

1.4.1. Rectangular Geometry

$$X_a(\Omega_1,\Omega_2) = \iint_{-\infty}^{\infty} x_a(t_1,t_2)\, e^{-j(\Omega_1 t_1 + \Omega_2 t_2)}\, dt_1\, dt_2$$

$$x_a(t_1,t_2) = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} X_a(\Omega_1,\Omega_2)\, e^{j(\Omega_1 t_1 + \Omega_2 t_2)}\, d\Omega_1\, d\Omega_2$$

Regaining $x_a(t_1,t_2)$ from

$$x[n_1,n_2] = x_a(n_1T_1, n_2T_2)$$

Now

$$\cancel{x[n_1,n_2] = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} X_a(\Omega_1,\Omega_2)\, e^{j}}$$

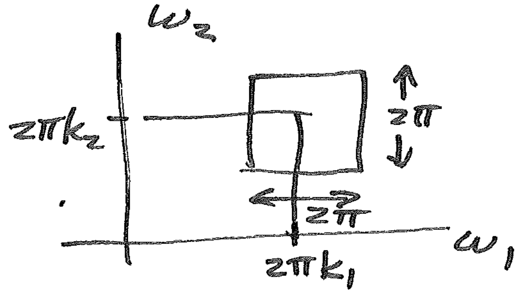
Set $\omega_1 = \Omega_1 T_1$, $\omega_2 = \Omega_2 T_2$

$$x[n_1,n_2] = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} X_a\left(\frac{\omega_1}{T_1}, \frac{\omega_2}{T_2}\right) e^{j(\omega_1 n_1 + \omega_2 n_2)} \times \frac{d\omega_1}{T_1}\frac{d\omega_2}{T_2}$$

Subdivide plane into squares:

$$-\pi + 2\pi k_1 \le \omega_1 < \pi + 2\pi k_1$$

$$-\pi + 2\pi k_2 \le \omega_2 < \pi + 2\pi k_2$$



$$x[n_1, n_2] = \frac{1}{(2\pi)^2} \sum_{k_1} \sum_{k_2} \iint_{SQ(k_1,k_2)} X_a\left(\frac{\hat{\omega}_1}{T_1}, \frac{\hat{\omega}_2}{T_2}\right)$$

$$\times e^{j(\hat{\omega}_1 n_1 + \hat{\omega}_2 n_2)} \frac{1}{T_1 T_2} \, d\hat{\omega}_1 d\hat{\omega}_2$$

Let

$$\omega_1 = \hat{\omega}_1 - 2\pi k_1$$

$$\omega_2 = \hat{\omega}_2 - 2\pi k_2$$

$$x[n_1, n_2] = \frac{1}{(2\pi)^2} \cancel{\frac{1}{T_1 T_2}} \frac{1}{T_1 T_2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \Bigg[\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty}$$

$$\times X_a\left(\frac{\omega_1 - 2\pi k_1}{T_1}, \frac{\omega_2 - 2\pi k_2}{T_2}\right)$$

$$\times e^{j(\omega_1 n_1 + \omega_2 n_2)} \, d\omega_1 d\omega_2 \, \overset{1}{\cancel{e^{+j 2\pi (k_1 n_1 + k_2 n_2)}}} \Bigg]$$

Result is inverse discrete transform. Thus

$$X(\omega_1, \omega_2) = \frac{1}{T_1 T_2} \sum_{k_1} \sum_{k_2} X_a\left(\frac{\omega_1 - 2\pi k_1}{T_1}, \frac{\omega_2 - 2\pi k_2}{T_2}\right)$$

Thus, the discrete spectrum is the replication of the continuous spectrum.

Further simplification if

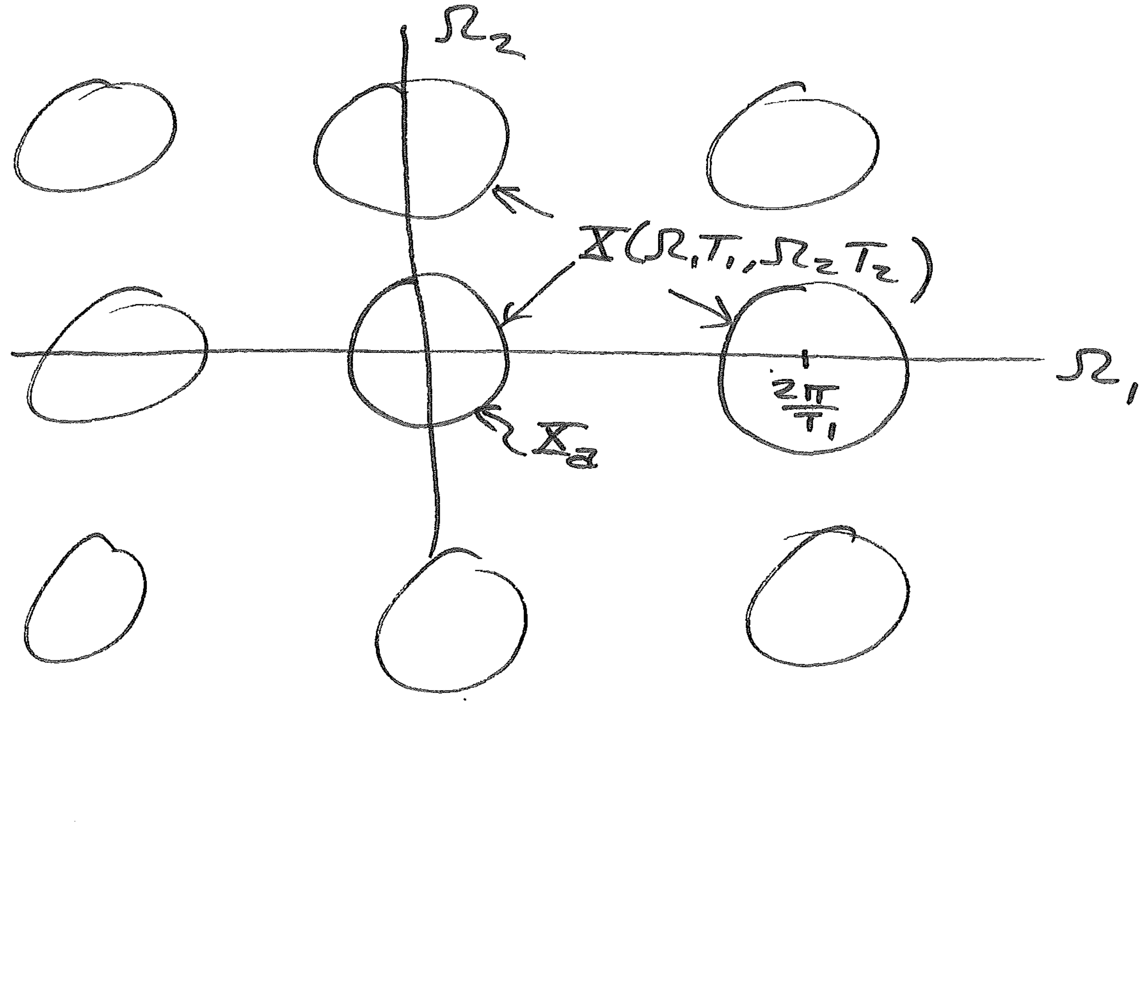
$$\underline{X}_a(\Omega,\Omega_2)=0 \qquad |\Omega_1|\geq \pi/T_1$$
$$|\Omega|_2\geq \pi/T_2$$

ie, x_a is Bandlimited. Then, over this rectangle:

$$\underline{X}(\omega_1\omega_2)=\frac{1}{T_1}\frac{1}{T_2}\underline{X}_a\left(\frac{\omega_1}{T_1},\frac{\omega_2}{T_2}\right)$$

or

$$\underline{X}(\Omega_1T_1,\Omega_2T_2)=\frac{1}{T_1T_2}\underline{X}_a(\Omega_1,\Omega_2)$$



Getting $x_a(t_1, t_2)$ from $X(\omega_1, \omega_2)$

$$x_a(t_1, t_2) = \frac{1}{(2\pi)^2} \iint X_a(\Omega_1, \Omega_2)\, e^{j(\Omega_1 t_1 + \Omega_2 t_2)}\, d\Omega_1\, d\Omega_2$$

$$= \frac{1}{(2\pi)^2} \int_{-W_1}^{W_1} \int_{-W_2}^{W_2} T_1 T_2\, X(\Omega_1 T_1, \Omega_2 T_2)\, e^{j(\Omega_1 t_1 + \Omega_2 t_2)}\, d\Omega_1\, d\Omega_2$$

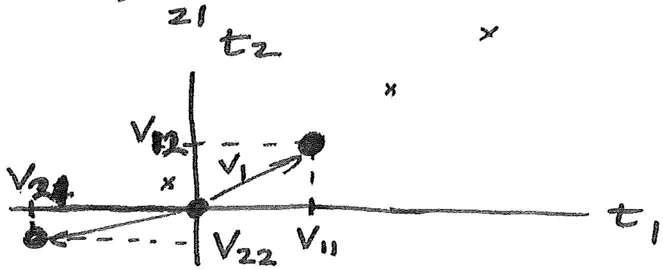
$W_1 = \pi / T_1$, $W_2 = \pi / T_2$

p. 39

1.4.2. Periodic Sampling with Arbitrary Sampling Geometries

Define sampling geometry:

$$\vec{V}_1 = \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} \qquad \vec{V}_2 = \begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix}$$



Sample locations at

$$t_1 = V_{11} n_1 + V_{12} n_2$$

$$t_2 = V_2 n_1 + V_{22} n_2$$

or:

$$\vec{t} = \underline{V}\,\vec{n} \quad ; \quad \underline{V} = \begin{bmatrix} V_{11} & V_{21} \\ V_{12} & V_{22} \end{bmatrix}$$

$$= [\vec{V}_1 \,\vdots\, \vec{V}_2]$$

= sampling matrix

$$\det \underline{V} \neq 0$$

Define samples:

$$x[\vec{n}] = x_a(\underline{V}\,\vec{n})$$

Relate xforms of $x[\vec{n}]$ & $x_a(\vec{t})$.

2-D Transform:

CONT
$$X_a(\vec{\Omega}) = \int_{\vec{t}} x_a(\vec{t})\, e^{-j\vec{\Omega}^T\vec{t}}\, d\vec{t}$$

$$x_a(\vec{t}) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} X_a(\vec{\Omega})\, e^{j\vec{\Omega}^T\vec{t}}\, d\vec{t}$$

DISCR
$$X(\vec{\omega}) = \sum_{\vec{n}} x[\vec{n}]\, e^{-j\vec{\omega}^T\vec{n}}$$

$$x[\vec{n}] = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi} X(\vec{\omega})\, e^{j\vec{\omega}^T\vec{n}}\, d\vec{\omega}$$

Thus:

$$x[\vec{n}] = x_a(\underline{V}\,\vec{n}) = \frac{1}{(2\pi)^2}\int_{\substack{-\infty \\ \vec{\Omega}}}^{\infty} X_a(\vec{\Omega})\, e^{j\vec{\Omega}^T\underline{V}\vec{n}}\, d\vec{\Omega}$$

Set $\vec{\omega} = \underline{V}^T\vec{\Omega} \Rightarrow \vec{\Omega} = (\underline{V}^T)^{-1}\vec{\omega},\ \vec{\Omega}^T = \vec{\omega}^T\,\underline{V}^{-1}$

$$\cancel{d\vec{\omega} = d\vec{\Omega} = (\underline{V}^T)^{-1} d\vec{\omega}}$$

Transformation Jacobian:

$$\begin{vmatrix} \frac{\delta\omega_1}{\delta\Omega_1} & \frac{\delta\omega_2}{\delta\Omega_2} \\ \frac{\delta\omega_2}{\delta\Omega_1} & \frac{\delta\omega_2}{\delta\Omega_2} \end{vmatrix} = \begin{vmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{vmatrix} = |\det \underline{V}|.$$

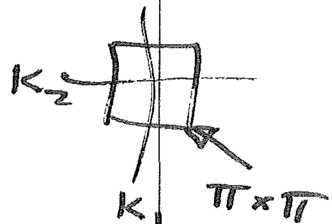
Thus:

$$x[\vec{n}] = \frac{1}{(2\pi)^2}\int_{\substack{-\infty \\ \vec{\omega}}}^{\infty} \frac{1}{|\det \underline{V}|}\, X_a(\underline{V}^{T^{-1}}\vec{\omega})\, e^{j\vec{\omega}^T\vec{n}}\, d\omega$$

Divide integrand into squares

$$\begin{cases} -\pi + 2\pi k_1 \le \omega_1 < \pi + 2\pi k_1 \\ -\pi + 2\pi k_2 \le \omega_2 < \pi + 2\pi k_2 \end{cases}$$

$SQ(k_1, k_2)$



Then

$$x[\vec{n}] = \frac{1}{(2\pi)^2} \sum_{\vec{k}} \int_{SQ(k_1,k_2)} \frac{1}{|\det \underline{V}|} X_a(\underline{V}^{T^{-1}} \hat{\vec{\omega}}) e^{j \hat{\vec{\omega}}^T \vec{n}} \, d\hat{\vec{\omega}}$$

Set $\hat{\vec{\omega}} = \vec{\omega} - 2\pi \vec{k}$

$$x[\vec{n}] = \frac{1}{(2\pi)^2} \sum_{\vec{k}} \int_{-\pi}^{\pi} \frac{1}{|\det \underline{V}|} X_a(\underline{V}^{T^{-1}}(\vec{\omega} - 2\pi \vec{k}))$$
$$e^{j\vec{\omega}^T \vec{n}} \underbrace{e^{-j 2\pi \vec{k}^T \vec{n}}}_{1} \cancel{\times e^{j\vec{\omega}^T \vec{n}}} \, d\vec{\omega}$$

~~Compare~~

Compare with

$$x[\vec{n}] = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} X(\vec{\omega}) e^{j\vec{\omega}^T \vec{n}} d\omega$$

Thus:

$$X(\vec{\omega}) = \frac{1}{|\det \underline{V}|} \cancel{X_a} \sum_{\vec{k}} X_a[\underline{V}^{T^{-1}}(\vec{\omega} - 2\pi \vec{k})]$$

Interpretation:

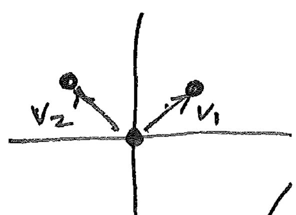
$\underline{U}$ = periodicity matrix in Fourier domain

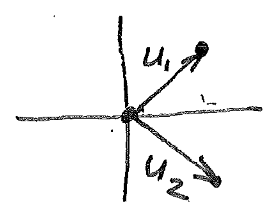
$= [\vec{u}_1 \mid \vec{u}_2]$

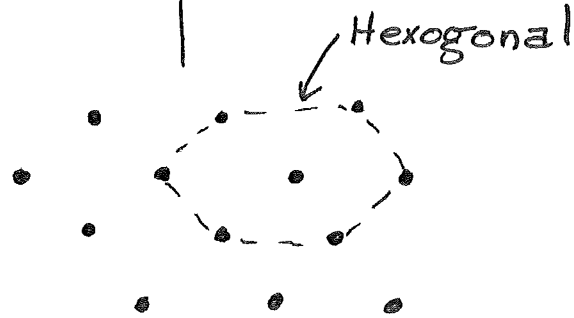
Clearly $\underline{X}(\underline{V}^T\vec{\Omega})$ is periodic wrt Ω

$$\underline{X}(\underline{V}^T(\vec{\Omega} + \underline{U}\vec{k})) = \underline{X}(\underline{V}^T\vec{\Omega} + 2\pi\vec{k})$$
$$= \underline{X}(\underline{V}^T\vec{\Omega})$$

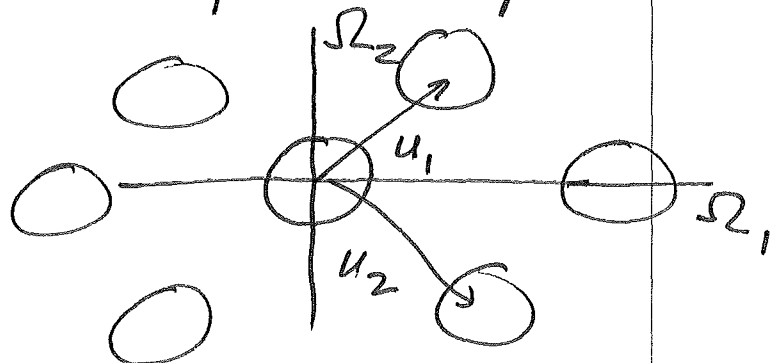
Ex $\quad \underline{V} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \underline{U} = \begin{bmatrix} \pi & \pi \\ \pi & -\pi \end{bmatrix}$





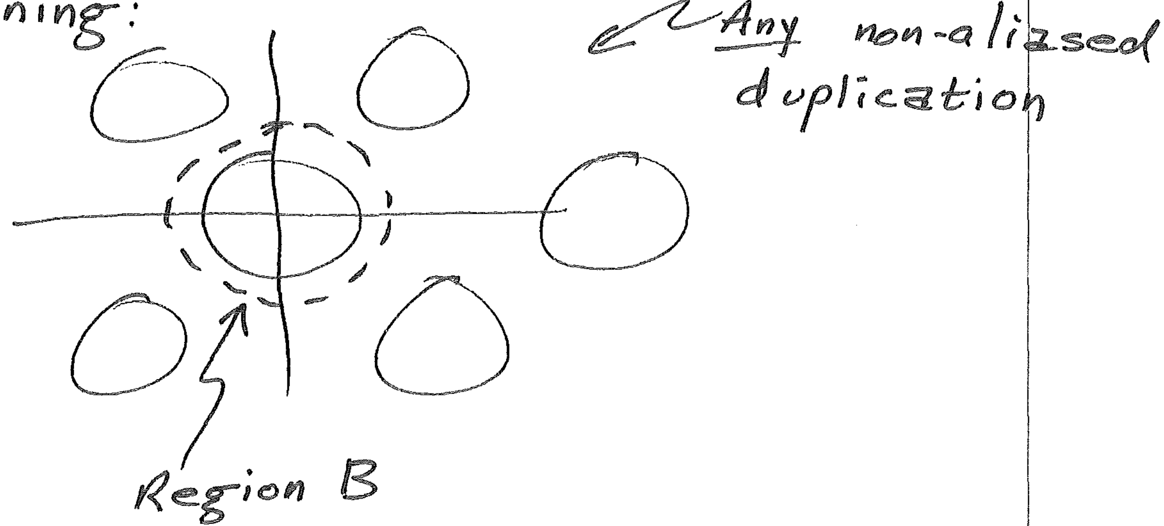


Spectrum duplicated:



If bandlimited, we can regain $x_a(\vec{t})$

Regaining:



$$X_a(\vec{\Omega}) = \begin{cases} |\det \underline{V}|\, X(\underline{V}^T\vec{\Omega}) & ; \ \vec{\Omega} \in B \\ 0 & ; \ o.w. \end{cases}$$

$\cancel{X_a}$

$$x_a(\vec{t}) = \frac{1}{(2\pi)^2} \int_{\vec{\Omega}} X_a(\vec{\Omega})\, e^{j\vec{\Omega}^T\vec{t}}\, d\vec{\Omega}$$

$$= \frac{|\det \underline{V}|}{(2\pi)^2} \int_B X(\underline{V}^T\vec{\Omega})\, e^{j\vec{\Omega}^T\vec{t}}\, d\vec{\Omega}$$

$$= \frac{|\det \underline{V}|}{(2\pi)^2} \int_B \left[\sum_{\vec{n}} x[\vec{n}]\, e^{-j(\underline{V}^T\vec{\Omega})^T\vec{n}} \right] e^{j\vec{\Omega}^T\vec{t}}\, d\Omega$$

$$= \sum_{\vec{n}} x[\vec{n}]\, \frac{|\det \underline{V}|}{(2\pi)^2} \int_B e^{-j\vec{\Omega}^T(\underline{V}\vec{n} - \vec{t})}\, d\vec{\Omega}$$

$$x(\vec{t}) = \sum_{\vec{n}} x[\vec{n}]\, f[\vec{t} - \underline{V}\vec{n}], \quad x[\vec{n}] = x[\underline{V}\vec{n}]$$

$$f(\vec{t}) = \frac{\det \underline{V}}{(2\pi)^2} \int_{\mathcal{B}} e^{-j\vec{\Omega}^T \vec{t}}\, d\vec{\Omega}$$

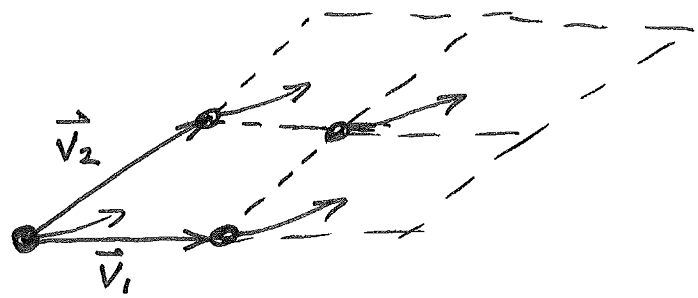
Genarize to M dimensions:

$$(2\pi)^2 \longrightarrow (2\pi)^M$$

Use same vector notation.

~~Sampling density.~~

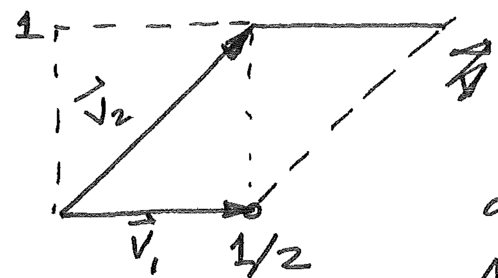
Sampling density:



One sample per parallelogram

Area of par $= \cancel{\frac{1}{|\det \underline{V}|}}\ |\det \underline{V}|$

<u>Ex</u>



$$\underline{V} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$$

$\det \underline{V} = 1/2$

Area $= |\det \underline{V}|$

$$\therefore \text{ sampling density} = \frac{1}{\det |\underline{V}|}$$

<u>or</u> Since $\underline{U} = 2\pi \underline{V}^{-1}$

$\underline{V} = 2\pi \underline{U}^{-1}$

$$\det V = (2\pi)^M \det \underline{U}^{-1} = \frac{(2\pi)^M}{\det \underline{U}}$$

Thus:

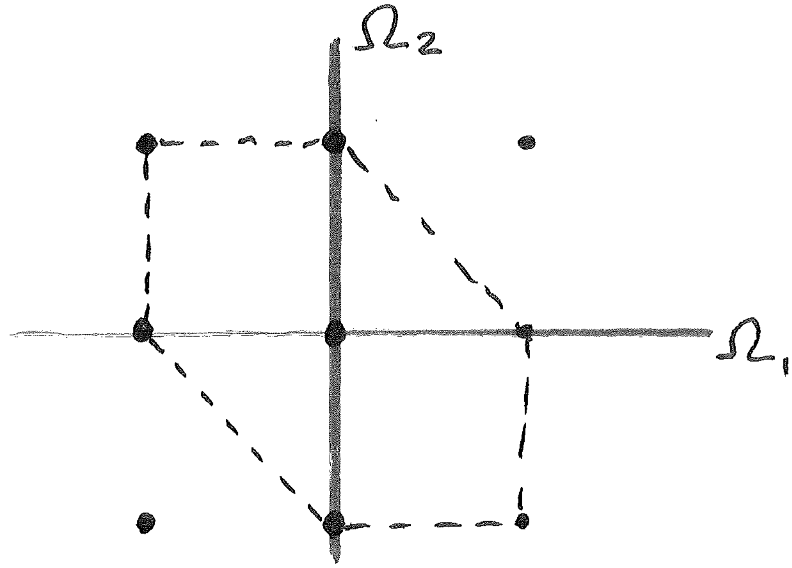
$$D = \frac{|\det \underline{U}|}{(2\pi)^M}$$

1.4.3. Comparison of Rectangular and Hexogonal Sampling

$$\text{Rect} \Longrightarrow \underline{V} = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \Longrightarrow \underline{U} = \begin{bmatrix} u_1 & 0 \\ 0 & u_2 \end{bmatrix}$$

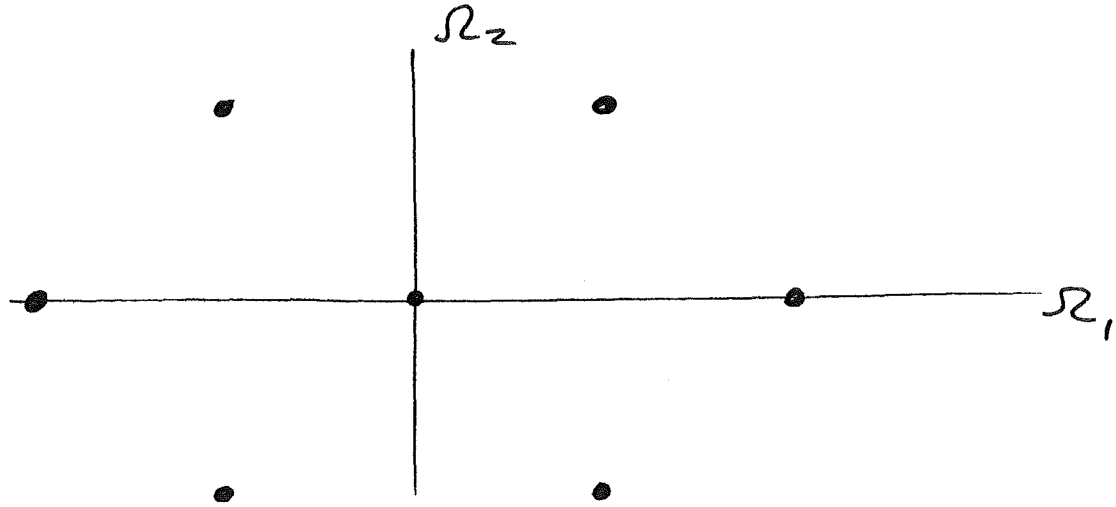
$$\text{Hex} \Longrightarrow \underline{V} = \begin{bmatrix} T_1 & T_1 \\ T_2 & -T_2 \end{bmatrix} \Longrightarrow \underline{U} = \begin{bmatrix} u_1 & u_1 \\ u_2 & -u_2 \end{bmatrix}$$

~~Then~~, for rect:



But, even rect rotated 45° is Hex

Hex:

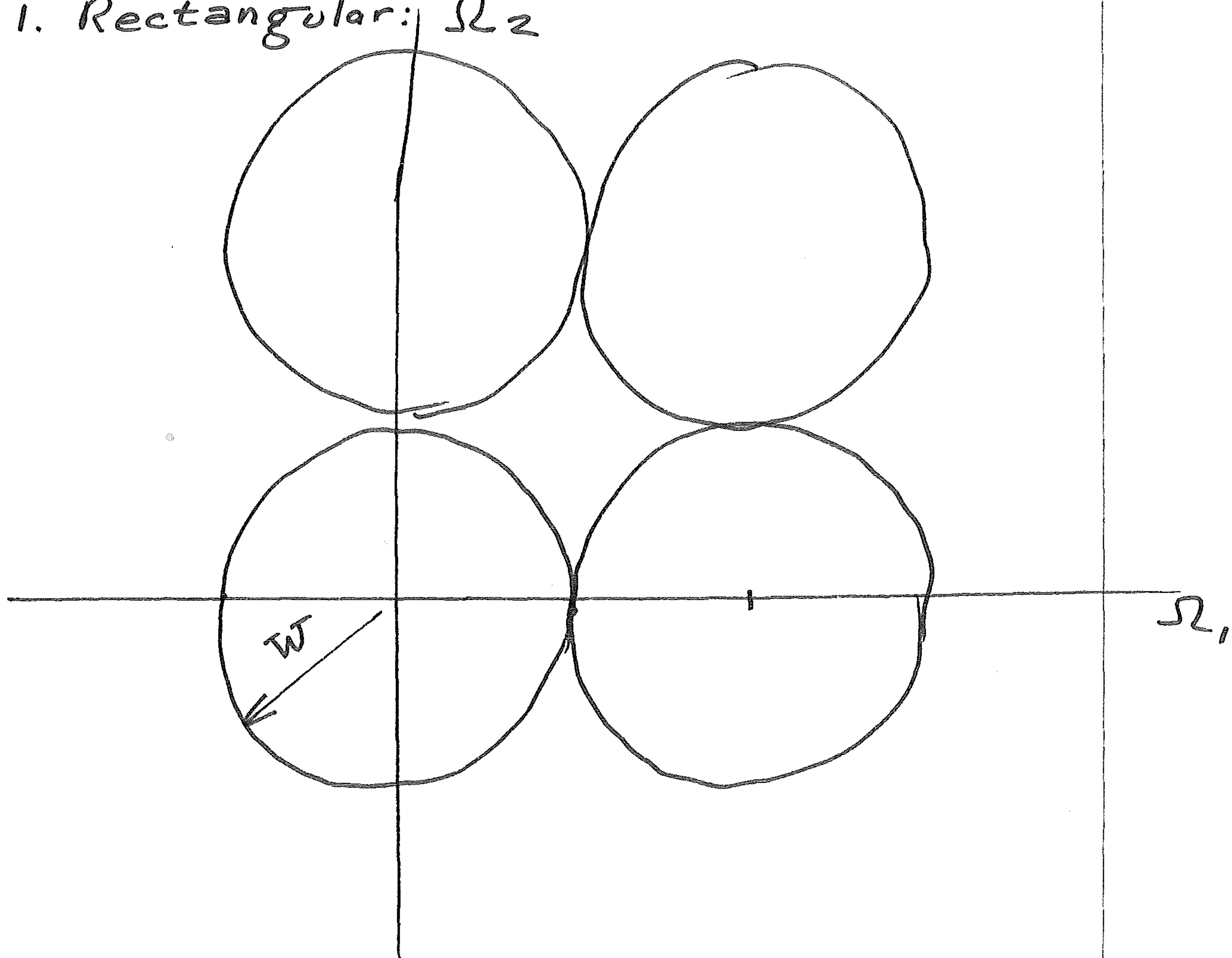


Patterns that fit hex on p.45.
(cover freq. plane)

In general, use $\underline{U}$ that will fill Ω plane. Some supports won't.
e.g. circle (in optics from circular lens)

Comparison:

1. Rectangular:

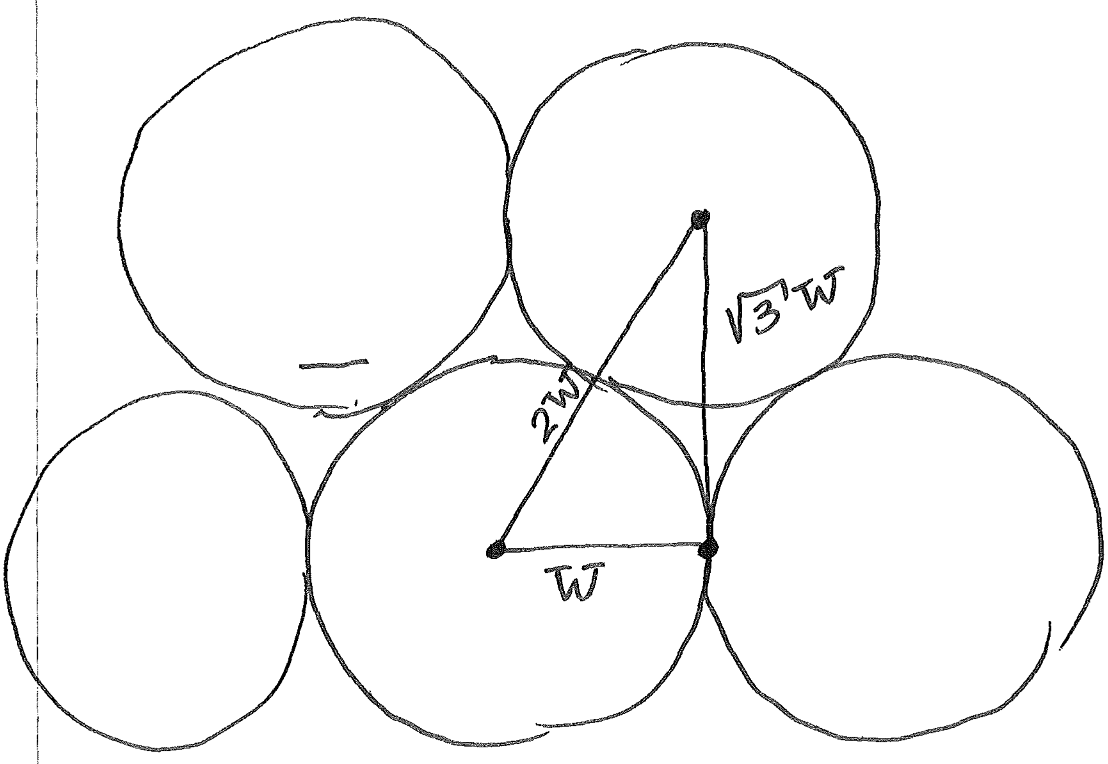


$$\underline{U} = \begin{bmatrix} 2W & 0 \\ 0 & 2W \end{bmatrix} \quad ; \det \underline{U} = (2W)^2$$

$$D = \left(\frac{W}{\pi}\right)^2$$

~~in~~ $\geq$ ~~M, D~~

2. HEXOGONAL



$$\underline{U} = \begin{bmatrix} w & w \\ \sqrt{3}\,w & -\sqrt{3}\,w \end{bmatrix}$$

$$|\det \underline{U}| = 2(\sqrt{3}\,w^2)$$

$$D = \frac{2\sqrt{3}\,w^2}{4\pi^2} = \frac{\sqrt{3}}{2}\left(\frac{w}{\pi}\right)^2$$

$< 1 \Rightarrow$ Hex has lower D.

(indeed, it's lowest)

OR, set $\omega = \underline{V}^T \vec{\Omega}$

$$\underline{X}(\underline{V}^T \vec{\Omega}) = \frac{1}{|\det \underline{V}|} \sum_{\vec{k}} X_a\left[\vec{\Omega} - V^{T^{-1}} 2\pi \vec{k}\right]$$

$$= \frac{1}{|\det \underline{V}|} \sum_{\vec{k}} X_a\left[\vec{\Omega} - \underline{U}\, \vec{k}\right]$$

$$\underline{U} = 2\pi \underline{V}^{T^{-1}}$$

<u>Example</u>: Rectangular

$$\underline{V} = \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} \qquad U = \begin{bmatrix} \frac{2\pi}{T_1} & 0 \\ 0 & \frac{2\pi}{T_2} \end{bmatrix}$$

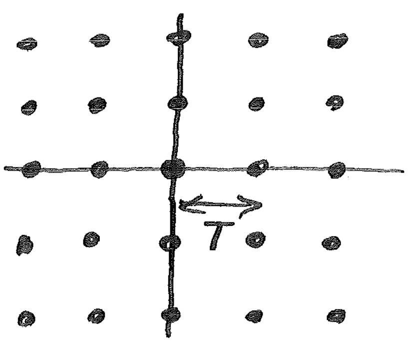
$$= \begin{bmatrix} 2W_1 & 0 \\ 0 & 2W_2 \end{bmatrix}$$

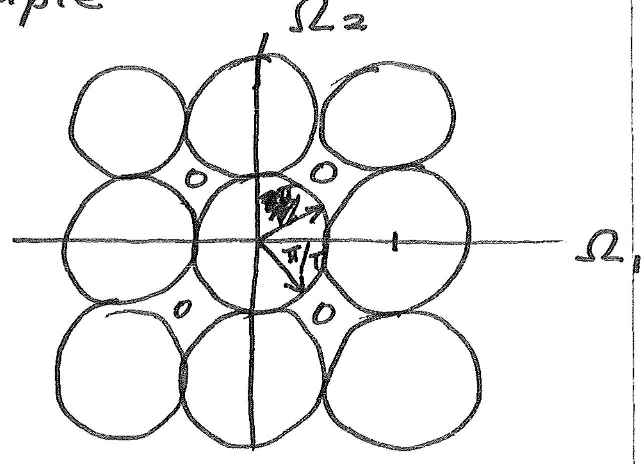
Same as before

Restoring Lost Samples:

If region of support is not connected (e.g.) circles) an arbitrary (finite) number of lost samples can be obtained from the remaining known samples.

Ex: 2-D 1 lost sample





$$x_a(t_1,t_2) = \sum_{n_1 n_2} x_a(n_1 T, n_2 T_2)\, \text{sinc}\left(\frac{t_1}{T} - n_1\right) \text{sinc}\left(\frac{t_2}{T} - n_2\right)$$

~~But, we can pass $x(t_1 t_2)$ thru a circular filter unaltered~~

$\cancel{x_a(t_1 t_2) =}$

~~Cannot regain a lost sample here. e.g. $t_1 = t_2 = 0$ Note cont~~

$$\cancel{x_a(0,0) = \sum_{n_1} \sum_{n_2} x_a(n T_1, n_2 T_2)\, \text{sinc}(n_1)\, \text{sinc}(n_2)}$$

$$\cancel{= x_a(0,0)} \quad \text{since } \text{sinc}(n_1) = \delta[n_1]$$

Note continuity:

$$x_a(k_1 T k_2 T) = \sum_{n_1} \sum_{n_2} x_a(n_1 T_1, n_2 T_2)\, \text{sinc}(k_1 - n_1)\, \text{sinc}(k_2 - n_2)$$

$$\text{since} = x_a(k_1 T, k_2 T$$

$$\text{sinc } m = \delta[m]$$

But, we can pass $x_a(t_1 t_2)$ thru a circular filter unaltered:

$$X_a(\Omega_1 \Omega_2) = \sum_{n_1} \sum_{n_2} x_a(n_1 T, n_2 T)$$

$$T_1 T_2 \, \mathrm{rect}\left(\frac{T_1 \Omega_1}{2\pi}, \frac{T_2 \Omega_2}{2\pi}\right)$$

$$\times e^{-j(\Omega_1 n_1 T_1 + \Omega_2 n_2 T_2)}$$

$$= T_1 T_2 \sum_{n_1} \sum_{n_2} x_a(nT, n_2 T) e^{-j(\Omega_1 n_1 T_1 + \Omega_2 n_2 T_2)}$$

$$; \ |\Omega_1| \leq \frac{\pi}{T}$$

$$|\Omega_2| \leq \frac{\pi}{T}$$

or:

$$\Omega = \sqrt{\Omega_1^2 + \Omega_2^2} \leq \frac{\pi}{T}$$

CIRCLE

Thus:

~~$x_a(n_1 T_1, n_2 T_2) = T_1 T_2 \int_O$~~

$$X(t_1 t_2) = \frac{T^2}{(2\pi)^2} \sum_{n_1} \sum_{n_2} x_a(n_1 T, n_2 T) \int_{\textcircled{O}} e^{+j[\Omega_1(t_1 - n_1 T) + \Omega_2(t_1 - n_2 T)]} d\Omega_1 d\Omega_2$$

~~$T_1 T_2$~~

But

$$\int_{Radius=1} e^{\pm j(\Omega_1 t_1 + \Omega_2 t_2)} d\Omega_1 d\Omega_2 = \frac{2\pi J_1(t)}{t} \quad ; t = \sqrt{t_1^2 + t_2^2}$$

$$\int_{Radius\ R} \Big[\qquad \Big] d\Omega_1 d\Omega_2 = R^{\cancel{2}} \frac{2\pi J_1(Rt)}{\cancel{R}t}$$

$$= \frac{2\pi R\, J_1(Rt)}{t}$$

Thus $(R = \frac{\pi}{T})$

$$x(t_1 t_2) = \frac{T^2}{(2\pi)^{\cancel{2}}} \sum_{n_1} \sum_{n_2} x_a(n_1 T, n_2 T)$$

$$\cancel{\frac{2\pi}{t}\left(\frac{\pi}{T}\right) J_1\left(\frac{\pi t}{T}\right)\left(\frac{\sqrt{(t_1 - nT)^2 + (t_2 - nT)^2}}{T}\right)}$$

$$\cancel{= \frac{T}{2t} \sum_{n_1} \sum_{n_2} x_a(n_1 T, n_2 T_2)\, J_1 (}$$

$$\frac{\cancel{2\pi}}{\sqrt{(t_1 - nT)^2 + (t_2 - nT)^2}} \cancel{\frac{\pi}{T}} J_1\left(\frac{\sqrt{(t_1 - n_1 T)^2 + (t_2 - n_2 T)^2}}{T}\right)$$

$$= \frac{1}{2} \sum_{n_1} \sum_{n_2} x_a(n_1 T_1, n_2 T_2) \cdot \underbrace{\frac{J_1\left(\sqrt{\left(\frac{t_1}{T} - n_1\right)^2 + \left(\frac{t_2}{T} - n_2\right)^2}\right)}{\sqrt{\left(\frac{t_1}{T} - n_1\right)^2 + \left(\frac{t_2}{T} - n_2\right)^2}}}_{jinc \sqrt{\left(\frac{t_1}{T} - n_1\right)^2 + \left(\frac{t_2}{T} - n_2\right)^2}}$$

$\frac{2\pi}{W}$

Samples now Dependent:

ex @ origin:

$$x(0,0) = \frac{1}{2} \sum_{n_1} \sum_{n_2} x_a(n_1 T, n_2 T)\, \text{jinc}\, \sqrt{n_1^2 + n_2^2}$$

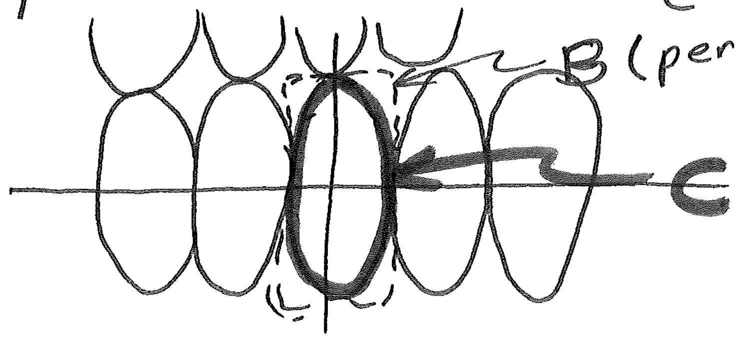
$$= \frac{1}{2} x_a(0,0)\, \text{jinc}\, 0$$

$$+ \frac{1}{2} \sum_{\substack{n_1, n_2 \\ (n_1,n_2) \neq (0,0)}} x_a(n_1 T, n_2 T)\, \text{jinc}\, \sqrt{n_1^2 + n_2^2}$$

OR

$$x_a(0,0) = \frac{1}{1 - \frac{1}{2}\text{jinc}\, 0} \left[\frac{1}{2} \sum_{(n_1,n_2) \neq (0,0)} x_a(n_1 T, n_2 T)\, \text{jinc}\, \sqrt{n_1^2 + n_2^2} \right]$$

Generalization to N dimensions. Loose M samples. Set of location $\mathcal{M}$



Recall:

$$f(\vec{t}) = \frac{|\det \underline{V}|}{(2\pi)^N} \int_{B} e^{j\vec{\Omega}^T \vec{t}} \, d\vec{\Omega}$$

$$x_a(\vec{t}) = \sum_{\vec{n}} x[\vec{n}] \, f(\vec{t} - \underline{V}\vec{n})$$

Define ~~two~~ regions, ~~B~~ & C. ~~B∈C~~ $\mathcal{C} \in \mathcal{B}$. $X_a \in \mathcal{C}$

$$x_a(\vec{t}) = \sum_{\vec{n}} x[\vec{n}] \, f_{C}(\vec{t} - \underline{V}\vec{n})$$

Loose M samples @ $\{\vec{k}_1, \vec{k}_2, \ldots \vec{k}_m\} = \mathcal{M}$ coordinates. Then:

$$\hat{x}_a(\vec{t}) = \left[\sum_{\vec{n} \in \mathcal{M}} + \sum_{\vec{n} \notin \mathcal{M}} \right] \hat{x}[\vec{n}] \, f_c(\vec{t} - \underline{V}\vec{n})$$

Evaluate at $\vec{t} = \underline{V}\vec{k}$ ~~; m = 1, 2, ..., M.~~ $\vec{k} \in \mathcal{M}$

Also, recall

$$x[\vec{n}] = x_a(\underline{V}\,\vec{n})$$

$$x_a(\underline{v}\vec{k}) = \Big[\sum_{\vec{n}\notin\mathcal{M}} + \sum_{\vec{n}\in\mathcal{M}}\Big] x_a(\underline{v}\vec{n})\, f_c\big(\underline{v}(\vec{k}-\vec{n})\big) \quad ; \vec{k}\in\mathcal{M}$$

Or:

$$\sum_{\vec{n}\in\mathcal{M}} x_a(\underline{v}\vec{n})\Big[\delta[\vec{n}-\vec{k}] - f_c\big(\underline{v}(\vec{k}-\vec{n})\big)\Big]$$

$$= \sum_{\vec{n}\notin\mathcal{M}} \overbrace{x_a(\underline{v}\vec{n})}^{\text{known}}\, f_c\big(\underline{v}(\vec{k}-\vec{n})\big)$$

$$\equiv g(\vec{k}) = \text{given}$$

M eqs & M unknowns. Solve for

$x_a(\underline{v}\vec{n})$; $\vec{n}\in\vec{k}$

$$\begin{array}{cccc} & \vec{n}\longrightarrow & & \\ \vec{k}\in\mathcal{M}\downarrow & \underbrace{\begin{bmatrix} \\ \delta[\vec{n}-\vec{k}] - f_c\big(\underline{v}(\vec{k}-\vec{n})\big) \\ \\ \end{bmatrix}}_{\text{singular if freq. plane is closed}} & \downarrow \vec{n}\in\mathcal{M}\begin{bmatrix} \\ x_a(\underline{v}\vec{n}) \\ \\ \end{bmatrix} = & \begin{bmatrix} g(\vec{k}_1) \\ \vdots \\ g(\vec{k}) \\ \\ \end{bmatrix}\downarrow\vec{k} \end{array}$$

Recall one lost sample:

$$\overline{n^2(\vec{0})} = \frac{f(\vec{0})\,\overline{\xi^2}}{1 - f(\vec{0})}$$

Here

$$\overline{\varphi^2(\vec{t})} = f(\vec{0})\,\overline{\xi^2}$$

Note, since $0 < 1 - f(\vec{0}) < 1$

$$\overline{n^2(\vec{0})} \geq \varphi^2(\vec{t})$$

DFT's:

$$X(\vec{k}) = \sum_{\vec{n} \in R_N} x[\vec{n}]\, e^{-j2\pi \vec{n} \underline{N}^{-1} \vec{k}}$$

$$x[n] = \frac{1}{|\det \underline{N}|} \sum_{\vec{k} \in R_N} X[\vec{k}]\, e^{j2\pi \vec{k} \underline{N}^{-1} \vec{n}}$$

Note Fourier Transform (rectangular)

$$X(\vec{\omega}) = \sum_{\vec{n} \in R_N} x[\vec{n}]\, e^{j \vec{\omega}^T \vec{n}}$$

$$\vec{\omega} \rightarrow \underline{N}^{-1} \vec{k} \quad \text{(notation abuse).}$$

2.2. Multidimensional DFT

Let $x[\vec{n}]$ be in $N_1 \times N_2 \times N_3 \cdots$ rect.
Periodic extension:

$$\tilde{x}[\vec{n}] = \sum_{\vec{r}} x[\vec{n} - \underline{N}\,\vec{r}] \quad ; \quad \underline{N} = \begin{bmatrix} N_1 & & & 0 \\ & N_2 & & \\ & & \ddots & \\ 0 & & & N_M \end{bmatrix}$$

or

$$x[\vec{n}] = \begin{cases} \tilde{x}[\vec{n}] & ; \ \vec{n} \in R_N = \text{rectangle} \\ 0 & ; \ \vec{n} \notin R_N \end{cases}$$

Consider DFS

$$\tilde{x}[\vec{n}] = \frac{1}{|\det \underline{N}|} \sum_{\vec{k} \in R_N} \tilde{X}(\vec{k})\, e^{j 2\pi \vec{n}^T \underline{N}^{-1} \vec{k}}.$$

$$= \frac{1}{N_1 N_2 \cdots N_M} \sum_{k_1=0}^{N_1-1} \cdots \sum_{k_M=0}^{N_M-1} \tilde{X}[k_1 \ldots k_M] \times e^{j 2\pi n_1 k_1/N_1} e^{j 2\pi n_2 k_2/N_2} \cdots e^{j 2\pi n_M k_M/N_M}$$

Fourier Series Coefficients

$$\tilde{X}(\vec{k}) = \sum_{\vec{n} \in R_N} \tilde{x}[\vec{n}]\, e^{-j 2\pi \vec{n}^T \underline{N}^{-1} \vec{k}}$$

(Proof as H.W.)

Can also write

$$\tilde{X}(\vec{k}) = \sum_{\vec{r}} X(\vec{k} - \underline{N}\,\vec{r}) \text{ since same form.}$$

$x[\vec{n}]$ & $X[\vec{k}]$ are DFT pairs on R_N

2.2.3. Multidimensional Circular Convolution

$$x[\vec{n}] \longleftrightarrow X[\vec{k}] \qquad h[\vec{n}] \longleftrightarrow H[\vec{k}]$$

$$? \longleftrightarrow Y[\vec{k}] = H[\vec{k}]\, X[\vec{k}]$$

Consider periodic function extension:

$$\tilde{x}[\vec{n}] \longleftrightarrow \tilde{X}[\vec{k}] \qquad \tilde{h}[\vec{n}] \longleftrightarrow \tilde{H}[\vec{k}]$$

$$? \longleftrightarrow \tilde{H}[\vec{k}]\, \tilde{X}[\vec{k}] = \tilde{Y}[\vec{k}]$$

Inverse DFS

$$\tilde{y}[\vec{n}] = \frac{1}{|\det \underline{N}|} \sum_{\vec{k} \in R_N} \tilde{H}[\vec{k}] \left[\tilde{X}[\vec{k}]\right] e^{j 2\pi \vec{k}^T \underline{N}^{-1} \vec{n}}$$

$$= \frac{1}{|\det \underline{N}|} \sum_{\vec{k} \in R_N} \tilde{H}[\vec{k}] \left[\sum_{\vec{m} \in R_N} \tilde{x}[\vec{m}] e^{j 2\pi \vec{k}^T \underline{N}^{-1} \vec{m}} \right] \times e^{-j 2\pi \vec{k}^T \underline{N}^{-1} \vec{n}}$$

$$= \frac{1}{|\det \underline{N}|} \sum_{\vec{m} \in R_N} \tilde{x}[\vec{m}] \times \sum_{\vec{k} \in R_N} \tilde{H}[\vec{k}] e^{j 2\pi \vec{k}^T \underline{N}^{-1} (\vec{n} - \vec{m})}$$

$$= \sum_{\vec{m} \in R_N} \tilde{x}[\vec{m}]\, \tilde{h}[\vec{n} - \vec{m}]$$

Define $y[\vec{n}] = \tilde{y}[\vec{n}]$; $\vec{n} \in R_N$

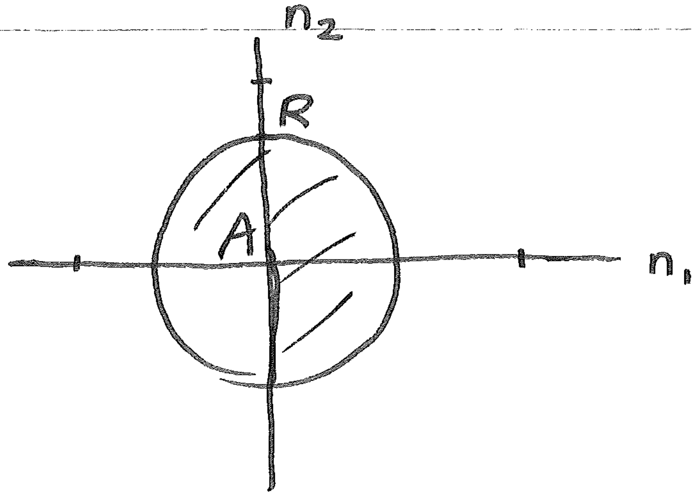
why

$$y[\vec{n}] = \sum_{\vec{m} \in R_N} x[\vec{m}]\, \tilde{h}[\vec{n} - \vec{m}] \; ; \; \vec{n} \in R_N$$

$$= \sum_{\vec{m} \in R_N} x[\vec{m}]\, h[(\vec{n} - \vec{m})_{\underline{N}}]$$ ← CIRCULAR CONVOLUTION

$$= x \circledast h \longleftrightarrow X H$$

Ex



$$h[n_1, n_2] = \frac{1}{(2\pi)^2} \iint_A e^{j(\omega_1 n_1 + \omega_2 n_2)} d\omega_1 d\omega_2$$

Let $\omega = \sqrt{\omega_1^2 + \omega_2^2}$; $\phi = \tan^{-1} \frac{\omega_2}{\omega_1}$

$$\Longrightarrow \omega_1 = \omega \cos\phi \qquad d\omega_1 d\omega_2 = \omega\, d\omega\, d\phi$$
$$\omega_2 = \omega \sin\phi$$

also let $\theta = \tan^{-1} n_2/n_1$ $\qquad n = \sqrt{n_1^2 + n_2^2}$

$$\cancel{\Rightarrow n_1 = \theta \cos\theta}$$

$$h[n_1, n_2] = \frac{1}{(2\pi)^2} \int_{\phi=0}^{2\pi} \int_{\omega=0}^{R} \omega e^{j\omega n \cos(\theta - \phi)} d\phi\, d\omega$$

$$= \frac{1}{(2\pi)^2} \int_{\substack{\cancel{\phi=0} \\ \omega=0}}^{\cancel{2\pi}} \omega \left[\int_{\phi=0}^{2\pi} e^{j\omega n \cos\phi} d\phi \right] d\omega$$

$$= \frac{1}{2\pi} \int_{\omega=0}^{R} \omega\, J_0(\omega n)\, d\omega$$

$$= \frac{R}{2\pi} \frac{J_1(Rn)}{n}$$

2.2.2. PROPERTIES OF DFT

Circular Shifts

$$\cancel{\tilde{x}[n_1,n_2] \longleftrightarrow \tilde{X}[k_1,k_2]}$$

$$\cancel{\tilde{y}[n_1,n_2] = \tilde{x}[n_1-m_1, n_2-m_2] \longleftrightarrow \tilde{X}[k_1,k_2}$$

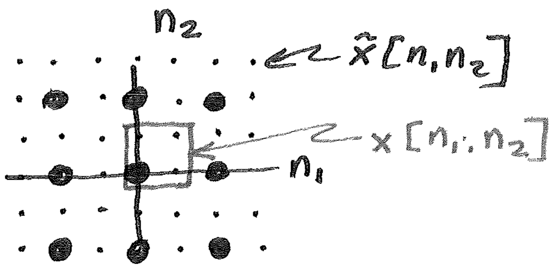
$$\tilde{x}[\vec{n}] \longleftrightarrow \tilde{X}[\vec{k}]$$

$$\tilde{y}[\vec{n}] = \tilde{x}[\vec{n}-\vec{m}] \longleftrightarrow \tilde{X}[\vec{k}]\, e^{-j2\pi \vec{k}\underline{N}^{-1}\vec{m}}$$

Define $y[\vec{n}] = \tilde{y}[\vec{n}] \in R_N$. Then $y[\vec{n}]$ is circular shift of $x[\vec{n}]$:

$$y[\vec{n}] = x[(\vec{n}-\vec{m})_{\underline{N}}] \quad ; \quad \vec{n} \in R_N$$

Ex:



Let $m_1 = m_2 = 1$

$$y[n_1,n_2] = x[(n_1-1)_2, (n_2-1)_2]$$

$$y[0,0] = x[(-1)_2, (-1)_2] = x[((-1)2+1)_2, ((-1)2+1)_2]$$

INT N_1

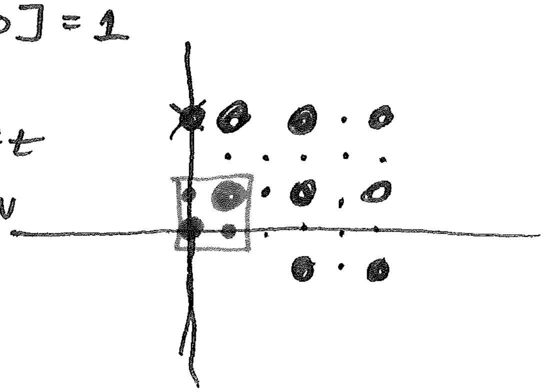
$$= x[1,1] = 0$$

$$y[0,1] = x[(-1)_2, (0)_2] = x[1,0] = 0$$

$$y[1,0] = x[0,1] = 0$$

$$y[1,1] = x[0,0] = 1$$

Net effect: shift $\tilde{x}$ and look in R_N



EE595

Multidimensional-signal sample dependency at Nyquist densities

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Received July 2, 1984; accepted September 23, 1985

When a multidimensional signal is uniformly sampled, its spectrum is replicated. If the signal is band limited and the replications (1) contain regions that are identically zero and (2) are not aliased, then the samples are dependent. Indeed, lost samples can be regained from those remaining. In dimensions greater than one, there are spectral regions of support for which this is the case even when sampling is performed at the Nyquist (minimum) density (e.g., a circular spectral region of support in two dimensions). When the known samples are perturbed by additive noise, lost-sample restoration noise levels in certain cases can be obtained by simple geometrical observations in the frequency domain. The results are specifically applied to coherent and incoherent optical images of objects of finite extent obtained from imaging systems with circular pupils.

1. INTRODUCTION

In one dimension, a band-limited signal's samples are independent when sampling is performed at the Nyquist rate. In higher dimensions, band-limited signal samples obtained at Nyquist (minimum) densities can display a strong dependence. Indeed, lost samples can be regained from those remaining. In the one-dimensional case, oversampling is required for sample dependency.[1,2]

The ability to restore lost samples of a multidimensional band-limited signal sampled at Nyquist density is determined solely by the shape of the support of the signal's spectrum. If the shape is such that replicated nonoverlapping versions can fill the space with no gaps, then Nyquist samples are independent. Otherwise, they are not.

An example of the former in two dimensions is a rectangle. A circle is an example of the latter. Any coherent or incoherent image of an object of finite extent obtained from an imaging system with a circular pupil has a spectrum with circular support.[3] Nyquist samples from such images are thus dependent, and lost samples can be evaluated from those remaining.

In this paper, after a brief review of the sampling theorem in N dimensions, we derive specific formulas for restoring lost samples in certain Nyqust sampled signals. The sensitivity of the restoration to additive noise is then presented. The results are fascinating interpretations of noise levels based on areas of regions of support. (Here and later, area refers to N-dimensional area; e.g., for $N = 3$, area refers to a volume). Applications to optical images are then addressed specifically.

2. PRELIMINARIES

Before stating the closed-form algorithm for lost-sample restoration, it is necessary to state the results of the N-dimensional sampling theorem for nonrectangular sampling geometry. Details of the theorem are admirably presented by Dudgeon and Mersereau[4] from Petersen and Middleton's initial treatment.[5]

N-Dimensional Sampling

Let $\{x(\mathbf{t})|\mathbf{t} = (t_1, t_2, \ldots, t_N)'\}$ denote an N-dimensional signal. (The prime is for vector or matrix transposition.) The corresponding spectrum is

$$X(\Omega) = \int_{\mathbf{t}} x(\mathbf{t}) \exp(-j\Omega'\mathbf{t})d\mathbf{t},$$

where $\Omega = (\Omega_1, \Omega_2, \ldots, \Omega_N)'$ and

$$\int_{\mathbf{t}} = \int_{t_1}\int_{t_2} \cdots \int_{t_N}.$$

The inverse transform is

$$x(\mathbf{t}) = \frac{1}{(2\pi)^N}\int_{\Omega} X(\Omega) \exp(j\Omega'\mathbf{t})d\Omega.$$

Let $\mathbf{V}$ be an $N \times N$ sampling matrix corresponding to the manner in which $x(\mathbf{t})$ is sampled. In general,

$$\mathbf{V} = [\mathbf{v}_1 \vert \mathbf{v}_2 \vert \ldots \vert \mathbf{v}_N],$$

where the $\mathbf{v}_n$'s are sampling vectors. For example, in Fig. 1, $N = 2$ and

$$\mathbf{V} = \begin{bmatrix} -1 & 2 \\ 3 & -2 \end{bmatrix}. \tag{1}$$

In general, the sampling density is

$$D = \frac{1}{|\det \mathbf{V}|}\,\frac{\text{samples}}{(\text{unit length})^N}.$$

For a specified $\mathbf{V}$, the sample signal is

$$\hat{x}(\mathbf{t}) = \sum_{\mathbf{n}} x(\mathbf{Vn})\delta_D(\mathbf{t} - \mathbf{Vn}), \tag{2}$$

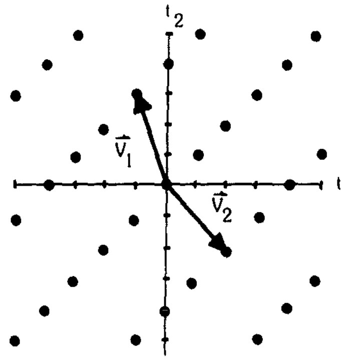


Fig. 1. Sampling geometry corresponding to the sampling matrix in Eq. (1).

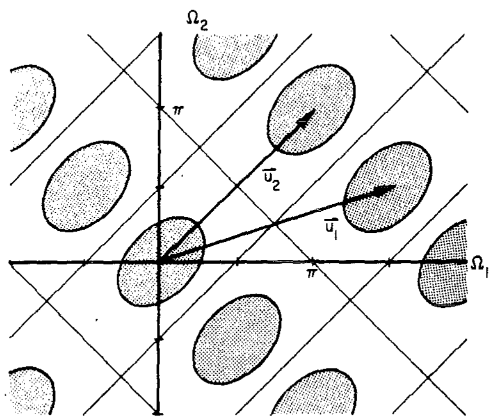


Fig. 2. Spectrum replication from the sampling geometry of Fig. 1.

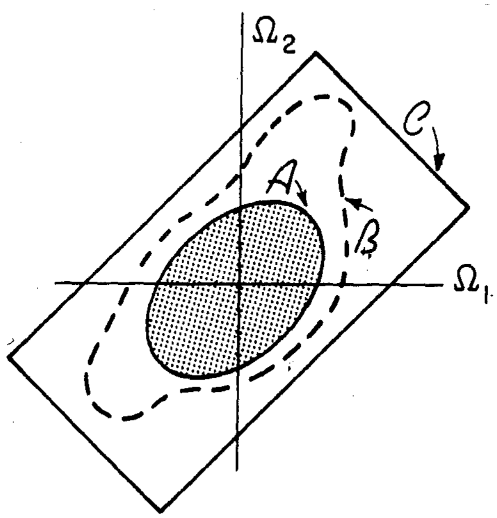


Fig. 3. One cell of Fig. 2. The region of integration, $\mathcal{B}$, must contain the spectral support region, $\mathcal{A}$, and must not infringe onto adjacent spectra. $\mathcal{C}$ is a cell region. The areas of the regions $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are A, B, and C, respectively.

where $\delta_D(\cdot)$ is the Dirac delta and $\mathbf{n} = (n_1, n_2, \ldots, n_N)'$. The spectrum of $\hat{x}(\mathbf{t})$ is the replication of the spectrum of $x(\mathbf{t})$:

$$\hat{X}(\Omega) = D \sum_k X(\Omega - \mathbf{U}\mathbf{k}), \tag{3}$$

where $\mathbf{U}$, the Fourier periodicity matrix, satisfies

$$\mathbf{U}'\mathbf{V} = 2\pi\mathbf{I}. \tag{4}$$

As we shall see, the geometry of the replication is dictated by $\{\mathbf{u}_n | n = 1, 2, \ldots, N\}$, where

$$\mathbf{U} = [\mathbf{u}_1 | \mathbf{u}_2 | \ldots | \mathbf{u}_N].$$

For our example in Eq. (1),

$$\mathbf{U} = \begin{bmatrix} \pi & 3\pi/2 \\ \pi & \pi/2 \end{bmatrix}.$$

Thus, if $X(\Omega_1, \Omega_2)$ were confined to be within the shaded ellipse at the origin in Fig. 2, then the corresponding $\hat{X}(\Omega_1, \Omega_2)$ would have the periodic structure shown.

For a given $\mathbf{V}$, there can exist a number of ways to separate $\hat{X}(\Omega)$ into periods. A period *cell*, when replicated, must fill the entire Ω plane. For a given $\mathbf{V}$, all cells will clearly have the same area. A possible cell for the example in Fig. 2 is the rotated rectangle shown in Fig. 3.

The N-Dimensional Sampling Theorem

An N-dimensional signal is band limited in the low-pass sense if its spectrum is identically zero outside of an N-dimensional hypersphere of finite radius. Then we can find a sampling matrix $\mathbf{V}$ such that the corresponding sample spectrum consists of nonoverlapping components. Under this condition, it is possible to regain $X(\Omega)$ from $\hat{X}(\Omega)$ in Eq. (3). We choose a region $\mathcal{B} \in \Omega$ that contains only the zeroth-order spectrum. Then

$$X(\Omega) = \hat{X}(\Omega)F(\Omega), \tag{5}$$

where

$$F(\Omega) = \begin{cases} |\det \mathbf{V}|; & \Omega \in \mathcal{B} \\ 0; & \Omega \notin \mathcal{B} \end{cases}.$$

An illustration for our running example is shown in Fig. 3. Note that $\mathcal{B}$ could correspond to a cell region $\mathcal{C}$ or the spectrum's region of support $\mathcal{A}$. To regain $x(\mathbf{t})$, we inverse transform Eq. (5) and obtain

$$x(\mathbf{t}) = \hat{x}(\mathbf{t}) * f(\mathbf{t}),$$

where the asterisk denotes N-dimensional convolution and

$$f(\mathbf{t}) = \frac{|\det \mathbf{V}|}{(2\pi)^N} \int_{\mathcal{B}} \exp(j\Omega'\mathbf{t})d\Omega. \tag{6}$$

Substituting Eq. (2) gives the desired interpolation formula:

$$x(\mathbf{t}) = \sum_{\mathbf{n}} x(\mathbf{V}\mathbf{n})f(\mathbf{t} - \mathbf{V}\mathbf{n}). \tag{7}$$

3. RESTORING LOST SAMPLES

In this section, we will show that an arbitrarily large but finite number of lost samples can be regained from those remaining for certain band-limited signals even when sampling is performed at the minimum density. The problem addressed is one of well-posed interpolation rather than ill-posed extrapolation.[6–9]

Let $\mathcal{M}$ denote a set of M integer vectors corresponding to the $\mathcal{M}$ lost-sample locations in an N-dimensional band-limited signal sampled in accordance with a sampling matrix, $\mathbf{V}$.

Theorem: If $x(\mathbf{t})$ is a band-limited signal and $\mathbf{V}$ is chosen to ensure that there is no aliasing between adjacent cells, then the missing samples can be regained from solution of the M equations:

$$\sum_{\mathbf{n}\in\mathcal{M}} \{\delta(\mathbf{k}-\mathbf{n}) - f[\mathbf{V}(\mathbf{k}-\mathbf{n})]\}x(\mathbf{Vn}) = \sum_{\mathbf{n}\notin M} x(\mathbf{Vn})f[\mathbf{V}(\mathbf{k}-\mathbf{n})]; \quad \mathbf{k}\in\mathcal{M} \tag{8}$$

assuming that the solution is not singular. [The Kronecker delta function, $\delta(\mathbf{n})$, is unity when $\mathbf{n} = \mathbf{O}$ and is zero otherwise.] The left-hand side of Eq. (8) contains the unknown samples. The right-hand side can be found from the known data.

Corollary: For a single lost sample at the origin, if $f(\mathbf{O}) \neq 1$,

$$x(\mathbf{O}) = [1 - f(\mathbf{O})]^{-1} \sum_{\mathbf{n}\neq\mathbf{O}} x(\mathbf{Vn})f(-\mathbf{Vn}). \tag{9}$$

This follows from Eq. (8) for $M = 1$ and $\mathcal{M}$ containing only the origin. Note that, by using Eq. (7), the signal's interpolation can be written directly void of the sample at the origin:

$$x(\mathbf{t}) = \sum_{\mathbf{n}\neq\mathbf{O}} x(\mathbf{Vn})[f(t - \mathbf{Vn}) + \{1 - f(\mathbf{O})\}^{-1}f(-\mathbf{Vn})f(\mathbf{t})].$$

Theorem Proof: We can write Eq. (7) as

$$x(\mathbf{t}) = \left(\sum_{\mathbf{n}\in\mathcal{M}} + \sum_{\mathbf{n}\notin\mathcal{M}}\right) x(\mathbf{Vn})f(\mathbf{t} - \mathbf{Vn}).$$

This expression can be evaluated at M points, and we can solve for $\{x(\mathbf{Vn})|\mathbf{n}\in\mathcal{M}\}$. Let these M points be the $\mathbf{t} = \mathbf{Vk}$, where $\mathbf{k}\in\mathcal{M}$:

$$x(\mathbf{Vk}) = \left(\sum_{\mathbf{n}\in\mathcal{M}} + \sum_{\mathbf{n}\notin\mathcal{M}}\right) x(\mathbf{Vn})f\{\mathbf{V}(\mathbf{k}-\mathbf{n})\}; \quad \mathbf{k}\in\mathcal{M}.$$

Rearranging gives Eq. (8).

Corollary: A sufficient condition for Eq. (8) to be singular is when the integration region, $\mathcal{B}$, is equal to a cell region, $\mathcal{C}$.

Proof: On a cell, the functions $\{\exp(j\boldsymbol{\Omega}'\mathbf{Vn})\}$ form an orthogonal basis set. From Eq. (6) with $\mathcal{B} = \mathcal{C}$ we have

$$f(\mathbf{Vn}) = \frac{|\det \mathbf{V}|}{(2\pi)^N}\int_{\mathcal{C}} \exp(j\boldsymbol{\Omega}'\mathbf{Vn})\mathrm{d}\boldsymbol{\Omega}.$$
$$= \delta(\mathbf{n}).$$

The left-hand side of Eq. (8) is thus zero and the resulting set of equations singular.

The restoration algorithm in this section alternatively could have been derived by a generalization of the iterative technique in Ref. 1. The treatment here, however, is more compact although maybe less intuitive. The results in Ref. 1 are equivalent to the $N = 1$ case. The same is true of Section 4 and Ref. 2.

4. NOISE SENSITIVITY

Our purpose here is to investigate the restoration algorithm's performance when inaccurate data are used.[2,10] In general, the algorithm becomes more unstable when (1) M increases and/or (2) the area corresponding to $\mathcal{B}$ increases with respect to that of $\mathcal{C}$. Indeed, restoration is no longer possible when $\mathcal{B} = \mathcal{C}$.

The restoration algorithm in Eq. (8) is linear. Let $\xi(\mathbf{t})$ denote a zero mean stochastic process. If $x(\mathbf{t})$ is uncorrelated with $\xi(\mathbf{t})$, then the use of $\{x(\mathbf{Vn}) + \xi(\mathbf{Vn})|\mathbf{n}\notin\mathcal{M}\}$ in Eq. (8) instead of $\{x(\mathbf{Vn})|\mathbf{n}\notin\mathcal{M}\}$ will result in $\{x(\mathbf{Vn}) + \eta(\mathbf{Vn})|\mathbf{n}\in\mathcal{M}\}$, where $\{\eta(\mathbf{Vn})|\mathbf{n}\in\mathcal{M}\}$ is the response to $\{\xi(\mathbf{Vn})|\mathbf{n}\notin\mathcal{M}\}$ alone:

$$\sum_{\mathbf{n}\in\mathcal{M}} [\delta(\mathbf{k}-\mathbf{n}) - f\{\mathbf{V}(\mathbf{k}-\mathbf{n})\}]\eta(\mathbf{Vn}) = \sum_{\mathbf{n}\notin\mathcal{M}} \xi(\mathbf{Vn})f\{\mathbf{V}(\mathbf{k}-\mathbf{n})\}. \tag{10}$$

The restoration noise, η, depends linearly on the data noise, ξ. Thus the cross correlation between these two processes and the autocorrelation of η can be determined from a given data noise autocorrelation.[11]

Out treatment will be limited to the case when a single sample is lost and the data noise is samplewise white, i.e.,

$$E[\xi(\mathbf{Vn})\xi^*(\mathbf{Vm})] = \overline{\xi^2}\delta(\mathbf{n}-\mathbf{m}), \tag{11}$$

where $\overline{\xi^2}$ is the data noise level (variance) and E denotes expectation. With no loss in generality, we place the lost sample at the origin, and Eq. (10) becomes

$$\eta(\mathbf{O}) = [1 - f(\mathbf{O})]^{-1} \sum_{\mathbf{n}\neq\mathbf{O}} \xi(\mathbf{Vn})f(-\mathbf{Vn}).$$

Taking the square of the magnitude, expectating, and using Eq. (11) gives

$$\overline{\eta^2(\mathbf{O})}/\overline{\xi^2} = [1 - f(\mathbf{O})]^{-2} \sum_{\mathbf{n}\neq\mathbf{O}} |f(-\mathbf{Vn})|^2, \tag{12}$$

where the restoration noise level is

$$\overline{\eta^2(\mathbf{O})} = E[|\eta(\mathbf{O})|^2].$$

The sum in Eq. (12) can be evaluated through Eq. (9) with $x(\mathbf{t}) = f^*(-t)$ [$=f(t)$ since $F(\Omega)$ is real]. The result is

$$\overline{\eta^2(\mathbf{O})}/\overline{\xi^2} = \frac{f(\mathbf{O})}{1 - f(\mathbf{O})}. \tag{13}$$

The result has a fascinating geometrical interpretation. From Eq. (6)

$$f(\mathbf{O}) = \frac{|\det \mathbf{V}|}{(2\pi)^N}\int_{\mathcal{B}} \mathrm{d}\boldsymbol{\Omega}.$$

But, with an illustration in Fig. 3,

$$B = \int_{\mathcal{B}} \mathrm{d}\boldsymbol{\Omega}$$
$$= \text{area of integration, } \mathcal{B}$$

and

$$C = \int_{\mathcal{C}} \mathrm{d}\boldsymbol{\Omega}$$
$$= \text{area of cell, } \mathcal{C}$$
$$= |\det \mathbf{U}|$$
$$= (2\pi)^N/|\det \mathbf{V}|,$$

where we have used Eq. (4). Thus Eq. (13) can be written as

Thus:

$$\mathcal{F}\left[f(\sqrt{\xi})\right] = j\omega\, \mathcal{F}\left[f_A(\sqrt{\xi})\right] \cdot \sqrt{\frac{d\pi}{\omega}}$$

and

$$f(\sqrt{\rho}) = -\frac{1}{\pi}\, \frac{d}{d\rho} f_A(\sqrt{\rho}) * \frac{\mu(-\rho)}{\sqrt{-\rho}}$$

$$= -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\frac{d}{d\xi} f_A(\sqrt{\xi})\, \mu(\xi - \rho)\, d\xi}{\sqrt{\xi - \rho}}$$

$$\rho = r^2$$

$$f(r) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\frac{d}{d\xi} f_A(\sqrt{\xi})\, \mu(\xi - r^2)\, d\xi}{\sqrt{\xi - r^2}}$$

$$\frac{d}{d\xi} f_A(\sqrt{\xi}) = \frac{1}{2\sqrt{\xi}}\, f_A'(\sqrt{\xi})$$

$$\xi = x^2 \Rightarrow d\xi = 2x\,dx$$

$$f(r) = -\frac{1}{\pi} \int_{0}^{\infty} \frac{\frac{1}{\cancel{2x}}\, f_A'(x)\, \mu(x^2 - r^2)\, \cancel{2x}\, dx}{\sqrt{x^2 - r^2}}$$

$$\mu(x^2 - r^2) = \mu(x - r) \Rightarrow x \geq r$$

$$f(r) = -\frac{1}{\pi} \int_{r}^{\infty} \frac{f_A'(x)\, dx}{\sqrt{x^2 - r^2}} \Leftarrow$$ INVERSE ABEL XFORM

$$=$$

$$\frac{\overline{\eta^2(\mathbf{O})}}{\xi^2} = \left(\frac{C}{B} - 1\right)^{-1}. \tag{14}$$

The restoration noise level is thus directly determined by the areas of the integration region for $f(\mathbf{t})$ and the area of a cell. Equation (14) is a strictly increasing function of B. Thus, for minimum restoration noise level, we choose $\mathcal{B} = \mathcal{A}$ = the region of support of the signal $x(t)$.

For Nyquist density sampling in one dimension, $\mathcal{A} = \mathcal{B} = \mathcal{C}$. In this case oversampling is required to restore lost samples.[1] For higher dimensions, the restoration capability is dependent on the region of support of the signal's spectrum. If the support is the shape of a cell (e.g., rectangular, hexagonal), then restoration is not possible at the Nyquist density.

Filtering

Samplewise white noise has a uniform spectral density and thus significant high-frequency energy. Once lost data have been restored, the data noise level can be reduced by filtering the result through $\mathcal{B}$ assuming that $B < C$. The noise level at the lost sample location remains the same.[2] The noise level at locations far removed from the lost-sample locations will asymptotically be the same as that for the filtered noisy samples if no data were lost. If $\xi(\mathbf{V}n)$ is zero mean and stationary, then after filtering, the process $\psi(\mathbf{V}n)$ is also stationary. If the data noise is white as in Eq. (11), its spectral density is uniform in $\mathcal{C}$. Thus if we filter the noise through $\mathcal{B}$, the resulting normalized noise level is

$$\overline{\psi^2}/\overline{\xi^2} = B/C. \tag{15}$$

(A more rigorous derivation is given in Appendix A.) To minimize, we clearly would choose $\mathcal{B} = \mathcal{A}$.

For a single lost sample in samplewise white noise, the ratio of the restoration noise level to that of data far removed is, after filtering through $\mathcal{B}$,

$$\frac{\overline{\eta^2(\mathbf{O})}}{\overline{\psi^2}} = \left[1 - \frac{B}{C}\right]^{-1}, \tag{16}$$

where we have used Eqs. (14) and (15). To minimize, we again would choose $\mathcal{B} = \mathcal{A}$. Note that Eq. (16) exceeds both unity and Eq. (14).

5. APPLICATION TO IMAGING SYSTEMS

An object of finite extent is imaged through a system with a circular pupil. If the monochromatic illumination is either coherent or incoherent, the image will have a spectrum with support inside a circle whose radius W is proportional to that of the pupil.

Nyquist Sampling of Optical Images

The Nyquist sampling density here is achieved when the circles in the frequency domain are densely packed as is shown at the top of Fig. 4. This corresponds to a sampling matrix

$$\mathbf{V} = \begin{bmatrix} T & -T \\ T/\sqrt{3} & T/\sqrt{3} \end{bmatrix},$$

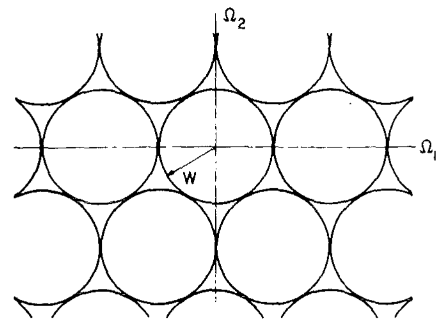
where $T = \pi/W$. The corresponding optimal sampling geometry, shown in Fig. 5, is thus hexagonal.[4]

Note, as is shown at the bottom of Fig. 4, that the area of $\mathcal{A}$ is less than that of $\mathcal{C}$. Thus, in the absence of noise, an arbitrary number of lost image samples can be restored from those (infinite number) remaining. For $\mathcal{B} = \mathcal{A}$, the interpolation function here is[3]

$$f(t_1, t_2) = \frac{W}{2\pi D} \frac{J_1[W(t_1{}^2 + t_2{}^2)^{1/2}]}{(t_1{}^2 + t_2{}^2)^{1/2}}.$$

Noise Effects

Here, we will numerically illustrate the effects of samplewise white noise on restoring a lost sample from an image that has a spectrum with circular support. Suboptimal rectangular sampling is considered first, followed by the optimal hexagonal case. Both cases are extended to higher dimensions.



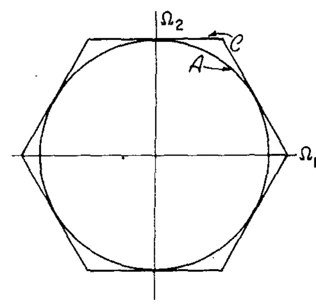


Fig. 4. Top, densely packed circles correspond to Nyquist sampling of images with spectra of circular support. Note the hexagonal structure. Bottom, a single hexagonal cell with inscribed circular spectrum support.

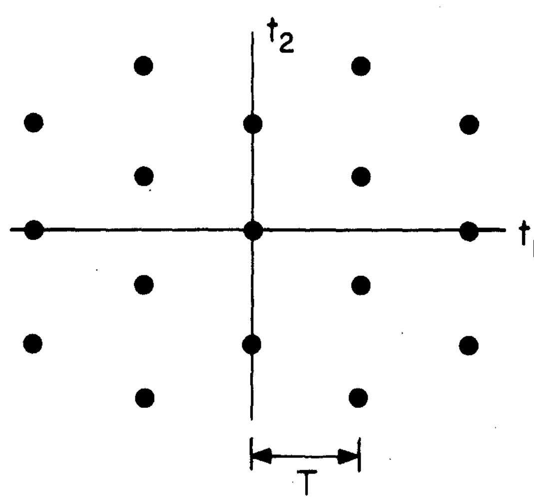


Fig. 5. Hexagonal sampling geometry required to pack circles densely as shown in Fig. 4.

Rectangular Sampling

If limited to rectangular sampling, minimum density sampling is accomplished by the sampling matrix

$$\mathbf{V} = \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix},$$

where $T = \pi/W$. The corresponding replicated spectra are shown at the top of Fig. 6. A single cell of this replication is shown on the bottom. The restoration noise level from Eq. (14) follows as

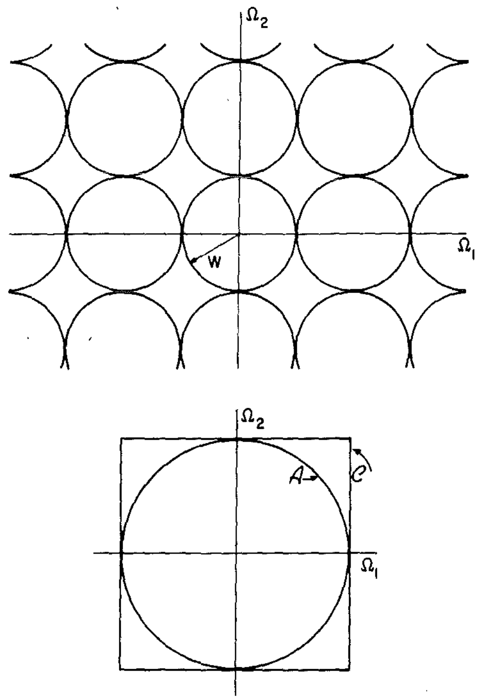


Fig. 6. Top, minimum density rectangular sampling of images with spectra of circular support yields circles packed as shown. Bottom, a single cell with inscribed circular spectrum support.

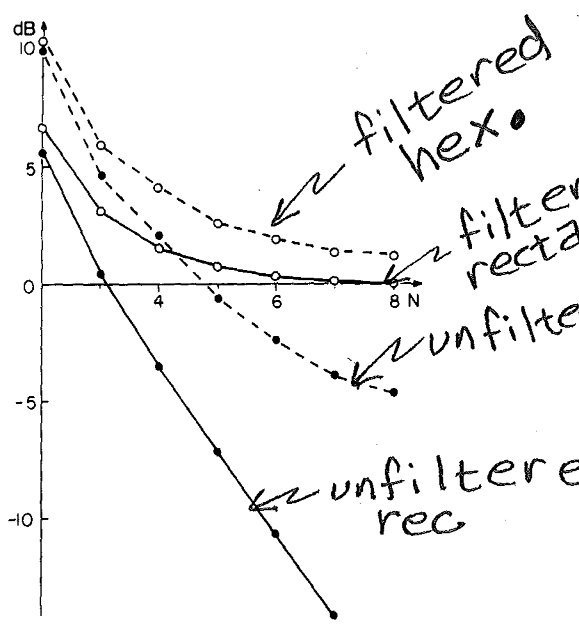


Fig. 7. Plots of $\overline{\eta^2(\mathbf{O})}/\xi^2$ (filled circles) and $\overline{\eta^2(\mathbf{O})}/\psi^2$ (open circles) in dB [10 $\log_{10}(\cdot)$]. The solid lines are for minimum density rectangular sampling and the dashed for Nyquist (hexagonal) sampling.

$$\frac{\overline{\eta^2(\mathbf{O})}}{\xi^2} = \left(\frac{4}{\pi} - 1\right)^{-1} \simeq 3.66 . \tag{17}$$

After filtering through the $\mathcal{A}$ circle, the ratio of the restoration noise level to data at points far removed from the origin is

$$\frac{\overline{\eta^2(\mathbf{O})}}{\psi^2} = \left[1 - \frac{\pi}{4}\right]^{-1} \simeq 4.66 , \tag{18}$$

where we have used Eq. (16) with $B = A = \pi W^2$. The lost-sample noise is thus 6.7 dB above the filtered data noise at infinity.

The results can easily be extended to higher dimensions. Assume that the spectrum has support within an N-dimensional hypersphere of radius W (Ref. 12):

$$A = \begin{cases} \dfrac{2^N \pi^{(N-1)/2} \left(\dfrac{N-1}{2}\right)!\, W^N}{N!} & \text{odd } N \\[2ex] \dfrac{\pi^{N/2}}{\left(\dfrac{N}{2}\right)!} W^N & \text{even } N \end{cases} . \tag{19}$$

For rectangular sampling, $C = (2W)^N$. The corresponding plots of $\overline{\eta^2(0)}/\xi^2$ and $\overline{\eta^2(\mathbf{0})}/\psi^2$ are shown as solid lines in Fig. 7.

Hexagonal Sampling

A single hexagonal cell is shown at the bottom of Fig. 7 for minimum density sampling. The area of the hexagon is

$$C = 2\sqrt{3}W^2.$$

Thus, from Eq. (14) for $B = \mathrm{A} = \pi W^2$

$$\frac{\overline{\eta^2(\mathbf{O})}}{\xi^2} = \left(\frac{2\sqrt{3}}{\pi} - 1\right)^{-1} \simeq 9.74,$$

and, similarly, from Eq. (16)

$$\frac{\overline{\eta^2(\mathbf{O})}}{\psi^2} = \left(1 - \frac{\pi}{2\sqrt{3}}\right)^{-1} \simeq 10.74 .$$

As one would expect, these values (~10 dB) are greater than those of the corresponding rectangular sampling cases in Eqs. (17) and (18).

In higher dimensions, Nyquist sampling would correspond to densely packed hyperspheres in the frequency domain. A table of the cell volume to circumscribed cubic volume is given by Dudgeon and Mersereau.[4] We can use this table in conjunction with Eq. (19) to generate the restoration noise level plots in Fig. 7 for Nyquist density sampling when the signal's spectrum support is a hypersphere. The plots are shown with broken lines and, as we would expect, exceed the corresponding rectangular sampling results.

6. CONCLUSIONS

We have shown that, in the absence of noise, an arbitrarily large but finite number of lost samples can be regained from those samples remaining under the conditions that (a) the data (with the lost samples) are not aliased and (b) there are sections in the sampled signal's spectrum that are identically zero. In dimensions greater than one, these conditions can apply even at Nyquist densities.

Noise analysis was performed for the case of one lost sample when the remaining data were corrupted by zero mean stationary white noise in terms of the sample. The resulting restoration noise levels are given by simple algebraic expressions involving various areas in the frequency domain. In all cases, minimum restoration noise level was achieved when the area of the support of the interpolation function's spectrum was at its minimum allowable value.

APPENDIX A

Here we derive Eq. (15). Let the samples be subjected to noise, $\xi(\mathbf{Vn})$, with autocorrelation as in Eq. (11). Then if $x(\mathbf{Vn}) + \xi(\mathbf{Vn})$ is used in Eq. (7) in lieu of $x(\mathbf{Vn})$, the result is $x(\mathbf{t}) + \psi(\mathbf{t})$, where

$$\psi(\mathbf{t}) = \sum_{\mathbf{n}} \xi(\mathbf{Vn}) f(\mathbf{t} - \mathbf{Vn}).$$

Squaring the magnitude of both sides and taking the expected value gives

$$\overline{\psi^2(\mathbf{t})} = \overline{\xi^2} \sum_{\mathbf{n}} |f(\mathbf{t} - \mathbf{Vn})|^2.$$

This sum can be evaluated using Eq. (7) with $x(\mathbf{t}) = f^*(\boldsymbol{\tau} - \mathbf{t})$:

$$f^*(\boldsymbol{\tau} - \mathbf{t}) = \sum_{\mathbf{n}} f^*(\boldsymbol{\tau} - \mathbf{Vn}) f(\mathbf{t} - \mathbf{Vn}).$$

For $\boldsymbol{\tau} = \mathbf{t}$ we obtain Eq. (15), recognizing that $\overline{\psi^2(\mathbf{t})} = \overline{\psi^2}$ is independent of $\mathbf{t}$.

Note that this result is a quantitative mesure of the tradeoff between sampling density and interpolation noise level.

ACKNOWLEDGMENTS

The author expresses his appreciation to Kwan F. Cheung for some valuable points and to Loretta Tollefson and Heidi Nurk for their assistance with the transcript and figures. Both reviewers kindly pointed out some further important references included here. One reviewer suggested the compact proof of the second corollary in Section 3, which was far superior to the author's initial effort.

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Noise Sensitivity of Lost Sample Restoration

Restoration Formula:

$$\sum_{\vec{n}\in\mathcal{M}} \left[\delta(\vec{n}-\vec{k}) - f(\underline{v}(\vec{k}-\vec{n}))\right] x_a(\underline{v}\,\vec{n}) = g(\vec{k}) \;;\; \vec{k}\in\mathcal{M} \qquad (1)$$

where

$$g(\vec{k}) = \sum_{\vec{n}\notin\mathcal{M}} x_a(\underline{v}\,\vec{n})\, f(\underline{v}(\vec{k}-\vec{n})) \;;\; \vec{k}\in\mathcal{M} \qquad (2)$$

The algorithm is linear. Thus, if $x_a(\underline{v}\vec{n}) + \xi(\underline{v}\vec{n})$ $(\vec{n}\notin\mathcal{M})$ is used as an input [$\xi(\vec{t})$ is noise], the restoration will be $x_a(\underline{v}\vec{n}) + n(\underline{v}\vec{n})$ $(\vec{n}\in\mathcal{M})$ where:

$$\sum_{\vec{n}\in\mathcal{M}} \left[\delta(\vec{n}-\vec{k}) - f(\underline{v}(\vec{k}-\vec{n}))\right] n(\underline{v}\,\vec{n}) = \varphi(\vec{k}) \;;\; \vec{k}\in\mathcal{M} \qquad (3)$$

$$\varphi(\vec{k}) = \sum_{\vec{n}\notin\mathcal{M}} \xi(\underline{v}\,\vec{n})\, f(\underline{v}(\underline{k}-\vec{n})) \;;\; \vec{k}\in\mathcal{M} \qquad (4)$$

Suppose $\xi(\vec{t})$ is zero mean with autocorrelation:

$$R_\xi(\vec{t};\vec{\tau}) = E\left[\xi(\vec{t})\,\xi(\vec{\tau})\right] \qquad (5)$$

Substituting (4) into (3):

$$\sum_{\vec{n}\in\mathcal{M}} \left[\delta(\vec{n}-\vec{k}) - f(\underline{v}(\vec{k}-\vec{n}))\right] n(\underline{v}\,\vec{n}) = \sum_{\vec{n}\notin\mathcal{M}} \xi(\underline{v}\vec{n})\, f(\underline{v}(\vec{k}-\vec{n})) \qquad (6)$$

Squaring both sidesand taking $E(\cdot)$ gives:

$$\sum_{\vec{n}\in\mathcal{M}} \sum_{m\in\mathcal{M}} \left[\delta(\vec{n}-\vec{k}) - f(\underline{v}(\vec{k}-\vec{n}))\right]\left[\delta(\vec{m}-\vec{k}) - f(\underline{v}(\vec{k}-\vec{m}))\right] \cdot R_n\{\underline{v}\,\vec{n};\underline{v}\,\vec{m}\} \qquad (7)$$

$$= \sum_{\vec{n}\notin\mathcal{M}} \sum_{\vec{m}\notin\mathcal{M}} f(\underline{v}(\vec{k}-\vec{n}))\, f(\underline{v}(\vec{k}-\vec{m}))\, R_\xi\{\underline{v}\,\vec{n};\underline{v}\,\vec{m}\}$$

SAMPLE WISE WHITE:

Assume:

$$R_\xi\{\underline{v}\vec{n};\underline{v}\vec{m}\} = \overline{\xi^2}\,\delta(\vec{n}-\vec{m}) \tag{8}$$

The hand side of (7) becomes:

$$\sum_{\vec{n}\notin\mathcal{M}}\sum_{\vec{m}\notin\mathcal{M}} f(\underline{v}(\vec{k}-\vec{n}))\, f(\underline{v}(\vec{k}-\vec{m}))\, R_\xi(\underline{v}\vec{n};\underline{v}\vec{m})$$

$$= \overline{\xi^2}\sum_{\vec{n}\notin\mathcal{M}} f^2(\underline{v}(\vec{k}-\vec{n})) \quad ; \quad \vec{k}\in\mathcal{M} \tag{9}$$

We can evaluate this sum using (1) and (2). Simply let

$$x_a(\vec{t}) = \overline{\xi^2} f(\underline{v}\ \vec{k}-\vec{t}) \tag{10}$$

The corresponding $g(\vec{k})$ in (2) is then equal to the sum in (9). We can evaluate $g(\vec{k})$ in (1) using (10). Thus

$$\overline{\xi^2}\sum_{\vec{n}\notin\mathcal{M}} f^2(\underline{v}(\vec{k}-\vec{n})) = \overline{\xi^2}\sum_{\vec{n}\in\mathcal{M}}\left[\delta(\vec{n}-\vec{k}) - f(\underline{v}(\vec{k}-\vec{n}))\right] * f(\underline{v}(\vec{k}-\vec{n})) \tag{11}$$

The ~~te~~ right hand term is a finite sum. Thus (7) becomes:

$$\sum_{\vec{n}\in\mathcal{M}}\sum_{\vec{m}\in\mathcal{M}}\left[\delta(\vec{n}-\vec{k}) - f(\underline{v}(\vec{k}-\vec{n}))\right] * \left[\delta(\vec{m}-\vec{k}) - f(\underline{v}(\vec{k}-\vec{m}))\right] R_\eta\{\underline{v}\vec{n};\underline{v}\vec{m}\}$$

$$= \overline{\xi^2}\sum_{\vec{n}\in\mathcal{M}}\left[\delta(\vec{n}-\vec{k}) - f(\underline{v}(\vec{k}-\vec{n}))\right] f(\underline{v}(\vec{k}-\vec{n})) \tag{1}$$

These are $M^2 * M^2$ linear equations with the same number of unknowns. The unknowns are the autocorrelations at the lost sample locations. Note that, although ξ is stationary [see (8)] that η will in general <u>not</u> be stationary.

A meaningful measure of the goodness of restoration is the variance

$$\overline{\eta^2(\vec{t})} = R_\eta(\vec{t};\vec{t}) \tag{13}$$

Then (12) becomes:

$$\sum_{\vec{n}\in\mathcal{M}} \left[\delta(\vec{n}-\vec{k}) - f(\underline{v}(\vec{k}-\vec{n}))\right]^2 \overline{\eta^2(\underline{v}\vec{n})}$$

$$= \overline{\xi^2} \sum_{\vec{n}\in\mathcal{M}} \left[\delta(\vec{n}-\vec{k}) - f(\underline{v}(\vec{k}-\vec{n}))\right] f(\underline{v}(\vec{k}-\vec{n})) \quad ; \ \vec{k}\in\mathcal{M} \tag{14}$$

SPECIAL CASE: 1. Sample Wise White Noise
2. One Sample Lost @ origin
$M=1$
$\vec{k}\in\mathcal{M} \Rightarrow \vec{k}=\vec{0}$

Then (14) becomes:

$$\left[\delta(\vec{k}) - f(\underline{v}\vec{k})\right]^2 \overline{\eta^2(\vec{0})}$$

$$= \overline{\xi^2}\left[\delta(\vec{k}) - f(\underline{v}\vec{k})\right] f(\underline{v}\vec{k}) \ ; \ \vec{k}=\vec{0}$$

or

$$\left[1 - f(\vec{0})\right]^{\cancel{2}} \overline{\eta^2(\vec{0})} = \cancel{\left[1 - f(\vec{0})\right]} f(\vec{0})\, \overline{\xi^2} \tag{15}$$

or

$$\frac{\overline{\eta^2(\vec{0})}}{\overline{\xi^2}} = \frac{f(\vec{0})}{1 - f(\vec{0})} \tag{16}$$

Since:

$$f(\vec{t}) = \frac{|\det \underline{V}|}{(2\pi)^M} \int_B e^{j\vec{\Omega}^T\vec{t}}\, d\vec{\Omega} \qquad (17)$$

we have:

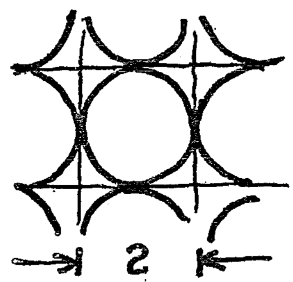
$$f(\vec{o}) = \frac{|\det \underline{V}|}{(2\pi)^M} \int_B d\vec{\Omega}$$

$$= \frac{\text{AREA OF BOUNDRY}}{\text{AREA OF CELL}} = \frac{B}{C} \qquad (18)$$

where:

$$|\det \underline{U}| = \left[\frac{|\det \underline{V}|}{(2\pi)^M}\right]^{-1} = \text{AREA OF CELL} \qquad (19)$$

Ex:



$B = \pi$
$C = 4$

Substituting into (16)

$$\overline{n^2(\vec{o})} / \overline{\xi^2} = \frac{\frac{B}{C}}{1 - \frac{B}{C}} = \frac{B}{C - B} = \left(\frac{C}{B} - 1\right)^{-1}$$

Example: M dimensional sphere spectrum support using rectangulat sampling.

$$B_M = \begin{cases} \dfrac{2^M \pi^{\frac{M-1}{2}} \left(\frac{M-1}{2}\right)! \rho^M}{M!} & ; M \text{ odd} \\ \rho^M \pi^{M/2} / \left(\frac{M}{2}\right)! & ; M \text{ even} \end{cases}$$

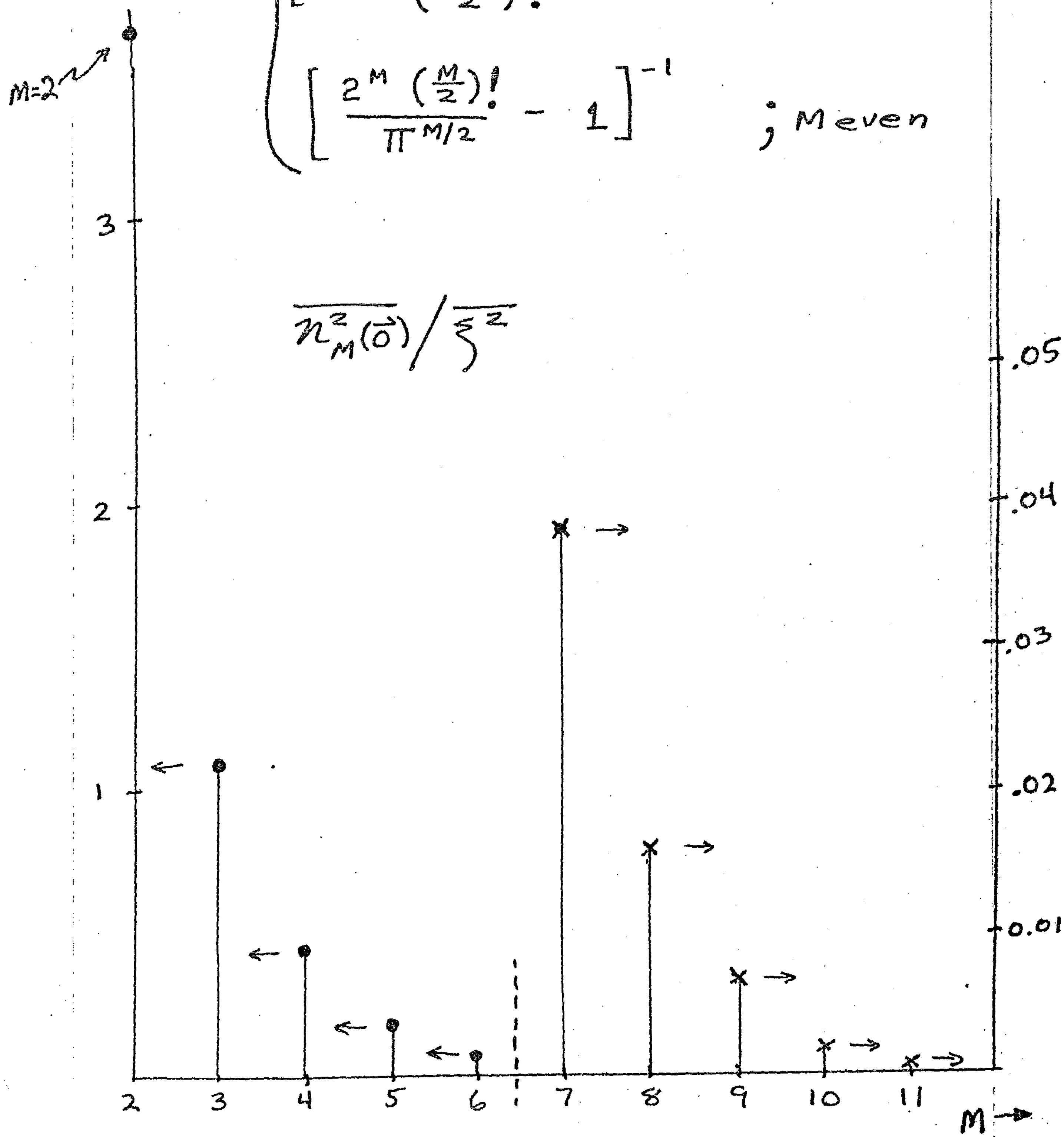
where ρ = radius. Also

$$C_M = (2\rho)^M$$

Thus:

$$\frac{\overline{n_M^2(\vec{0})}}{\overline{\xi^2}} = \left[\frac{C_M}{B_M} - 1\right]^{-1}$$

$$= \begin{cases} \left[\dfrac{M!}{\pi^{\frac{M-1}{2}}\left(\frac{M-1}{2}\right)!} - 1\right]^{-1} & ; \text{ M odd} \\ \left[\dfrac{2^M \left(\frac{M}{2}\right)!}{\pi^{M/2}} - 1\right]^{-1} & ; \text{ M even} \end{cases}$$



CHAPT 3

2-D FIR Filters

$$y[\vec{n}] = \sum_{\vec{k}} h[\vec{k}]\, x[\vec{n}-\vec{k}]$$

If $h[\vec{k}]$ has finite # of points, filter is FIR.

Freq. response:

$$H(\vec{\omega}) = \sum_{\vec{k}} h[\vec{k}]\, e^{-j\vec{\omega}^T\vec{k}}$$

If $h[\vec{k}] = h^*[-\vec{k}]$, H is real & visa versa (zero phase)

Proof: $H(\omega)$ real

~~$h[\vec{n}$ $H(\vec{\omega}) = \frac{1}{(2\pi)^M}\int_{\pi}$~~

set $\vec{k} \to -\vec{k}$

$$\left(\sum_{-\vec{k}=\vec{k}} h[-\vec{k}]\, e^{j\vec{\omega}^T k}\right)^*$$

$$= \sum_{\vec{k}} \underbrace{h^*[-\vec{k}]}_{= h[k]}\, e^{-j\vec{\omega}^T k}$$

GOOD for

3.2.1. Direct Convolution

for real zero phase filter:

$$h[\vec{k}] = h[-\vec{k}]$$

Operation reduction:

$$y[\vec{n}] = \sum_{\vec{k}=-\vec{N}}^{\vec{N}} h[\vec{k}]\, x[\vec{n}-\vec{k}]$$

where $h[\vec{k}] = 0$ outside of $2N_1 \times 2N_2 \times \ldots \times 2N_M$ box centered @ origin. In 2-D

$4N_1N_2$ ops

$$y[n_1, n_2] = \sum_{k_1=-N_1}^{N_1} \sum_{k_2=-N_2}^{N_2} h[k_1, k_2]\, x[n_1-k_1, n_2-k_2]$$

$$= \sum_{k_1=-N_1}^{N_1} \Big[h[k_1, 0]\, x[n_1-k_1, n_2] + \sum_{k_2=1}^{N_2} h[k_1, k_2] \big\{ x[n_1-k_1, n_2-k_2] + x[n_1-k_1, n_2+k_2] \big\}$$

$$= h[0,0]\, x[n_1, n_2]$$
$$+ \sum_{k_1=1}^{N} h[k_1, 0] \big\{ x[n_1-k_1, n_2] + x[n_1+k_1, n_2] \big\}$$
$$+ \sum_{k_2=1}^{N_2} h[0, k_2] \big\{ x[n_1-k_1, n_2-k_2] + x[n_1-k_1, n_2+k_2] \big\}$$
$$+ \sum_{k_1=-}$$

etc

Good for
h low order

3.22. DFT Implementations

$$y[\vec{n}] = \underbrace{x[\vec{n}]}_{N_1} * \underbrace{h[\vec{n}]}_{N_2} \qquad \text{(also Finite input)}$$

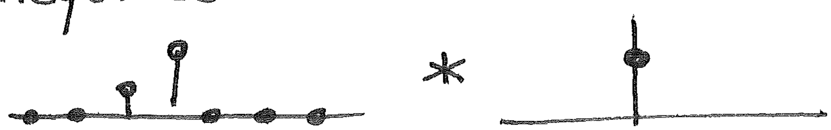
Can use DFT's

$$Y[\vec{k}] = X[\vec{k}]\, H[\vec{k}]$$

<u>But</u>, we get $x[\vec{n}] \circledast h[\vec{n}] \neq x[\vec{n}] * h[\vec{n}]$

How do we make equal? Pack with zeros.

~~Requires~~



Then $\circledast = *$ (elaborate)

Comp. ~~con~~ ineff.
Better than conv
for high order
(Lots of memory)

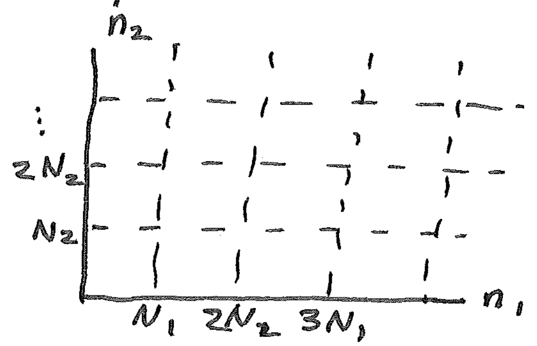
3.2.3. BLOCK CONVOLUTION (Large Range)

~~Low Filter Order ⇒ Convolution method more efficient~~

~~High Order ⇒ DFT more~~

~~Compromise: BLOCK CONVOLUTION~~

Overlap and Add:



divide input into blocks

$$x_{k_1,k_2}[n_1,n_2] = \begin{cases} x[n_1,n_2] \; ; & kN_1 \le n_1 \le (k_1+1)N_1 \\ & k_2N_2 \le n_2 \le (k_2+1)N_2 \\ 0 \; ; & \text{otherwise} \end{cases}$$

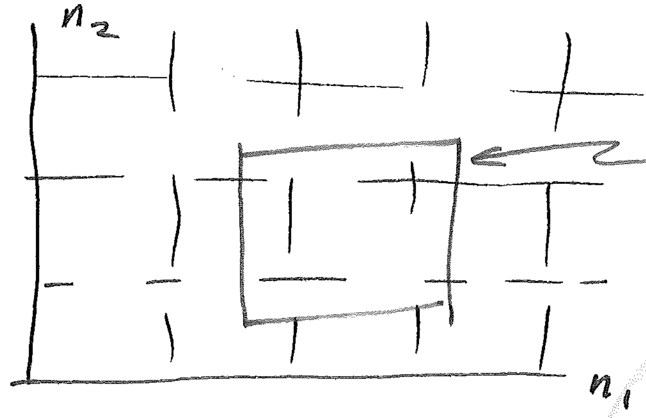
$$x[n_1,n_2] = \sum_{k_1}\sum_{k_2} x_{k_1,k_2}[n_1,n_2]$$

Then $y[n_1,n_2] = x[n_1,n_2] ** h[n_1,n_2]$

$$= \left(\sum_{k_1,k_2} (x_{k_1,k_2} ** h\right)$$

$$= \sum_{k_1}\sum_{k_2} y_{k_1,k_2}$$

Will be overlap. Since FIR, will be finite

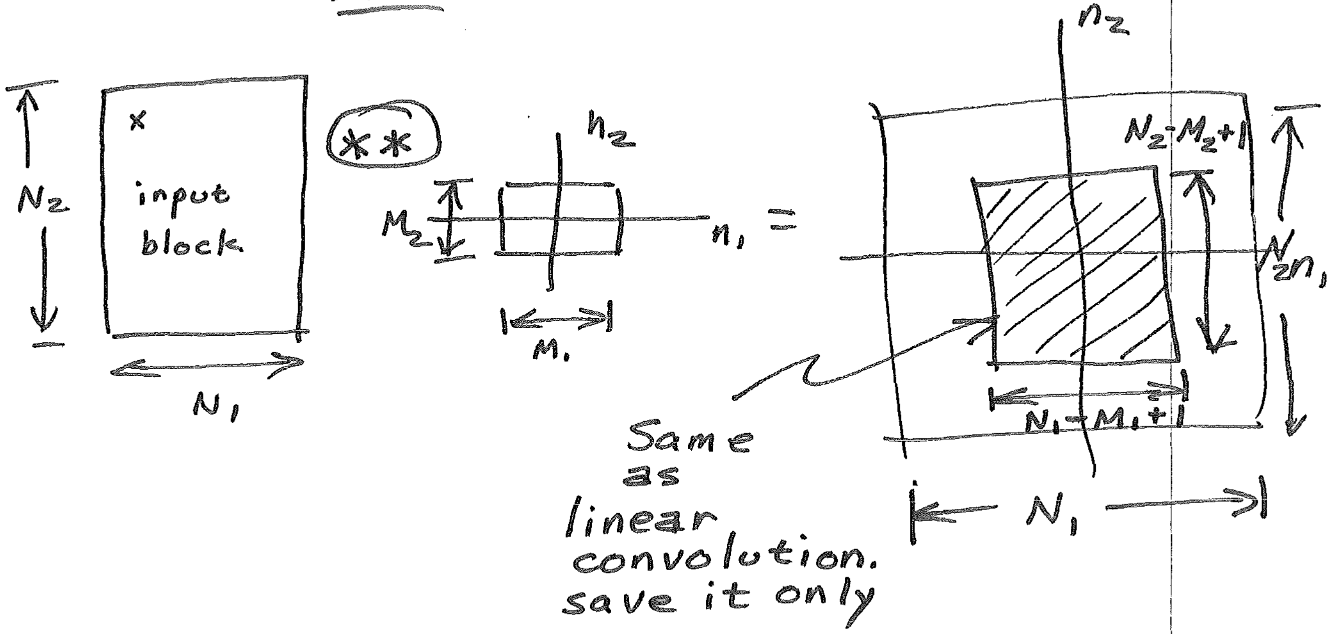


Rectangular h

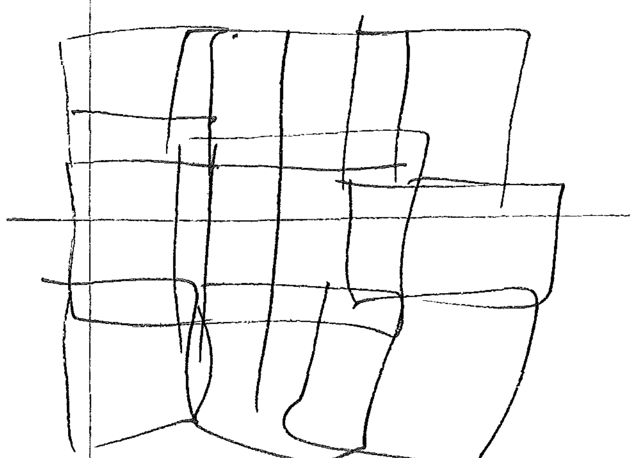
Use DFT's to compute $y_{k_1k_2}$'s

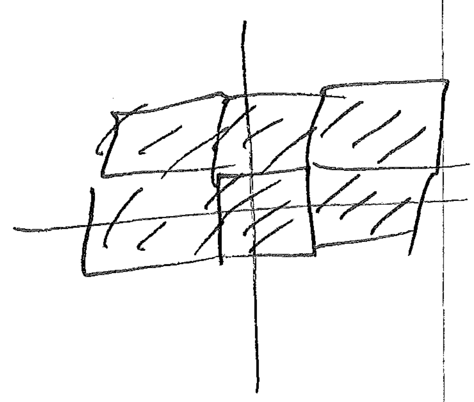
Note: Must zero pack to avoid circular convolution

OVERLAP AND SAVE (circular convolution)



Thus: ~~Oversample inp~~
Overlap Sample input (elaborate)





3.3. Design of FIR filters using windows

3.3.1. Method description

Wish to have FIR filter with freq. response $I(\vec{\omega})$ and impulse response ~~$i[n_1,n_2]$~~ $i[\vec{n}]$ but, is restricted to region R. ~~Define~~ Use

$$h[\vec{n}] = i[\vec{n}]\, w[\vec{n}] \;;$$

$w[\vec{n}]$ (and thus $h[\vec{n}]$) are confined to R.

Q: How close is $H(\vec{\omega})$ to $I(\vec{\omega})$?

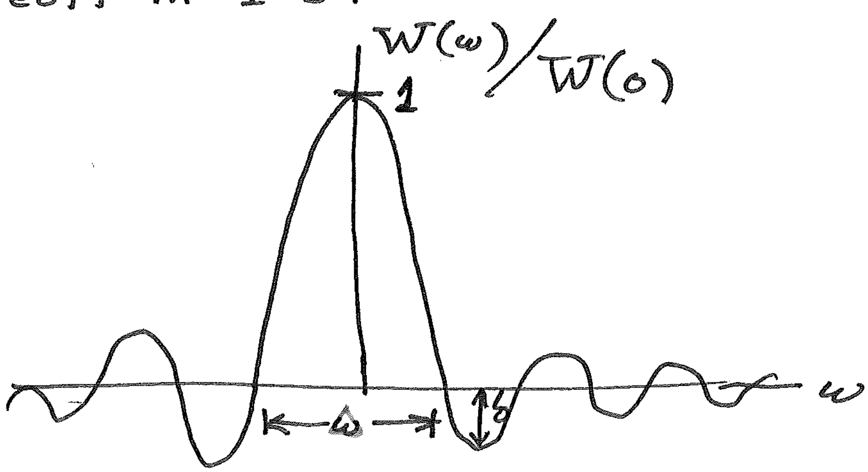
$$H(\vec{\omega}) = \frac{1}{(2\pi)^M} \int_{\#} I(\vec{\Omega})\, W(\vec{\omega} - \vec{\Omega})\, d\vec{\Omega}$$

would be exact if

$$W(\vec{\omega}) = (2\pi)^M \delta(\vec{\omega})$$

But this can't be. Since $w[\vec{n}] \in R_N$, W fills entire plane (elaborate).

Tradeoff in 1-D:

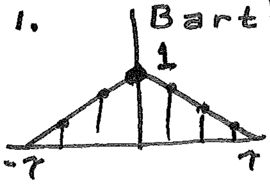


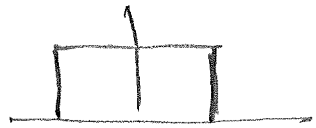
as $\omega\downarrow$, $\delta\uparrow$.

$\exists$ a number of "good" 1-D windows $(w(0) = 1)$

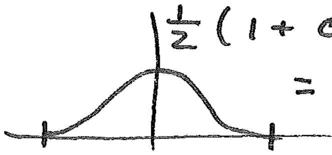
$$W(t) = w(t)\,\Pi\left(\frac{t}{2\tau}\right)$$

1. Bartlett





2. Tukey (Hanning)

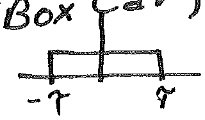


$$\frac{1}{2}\left(1 + \cos\frac{\pi}{\tau}t\right) = \cancel{\sin^2} \cos^2\frac{2\pi t}{2\tau}$$
$$= \cos^2 \pi t/2\tau$$

3. Hamming $\quad 0.54 + 0.46 \cos\frac{\pi t}{\tau}$

4. Parzen:

5. Rectang (Box Car)



6. Kaiser

$$w(t) = \begin{cases} \dfrac{I_0\left(\alpha\sqrt{1 - \left(\frac{t}{\tau}\right)^2}\right)}{I_0(\alpha)} & ; \ |t| < \tau \\ 0 & ; \ \text{o.w.} \end{cases}$$

Extension to ~~two~~ M-dimensions

1. ~~Outter~~ Outer product:

$$W_R(\vec{n}) = \prod_{m=1}^{M} w(n_m)$$

2. Rotated window:

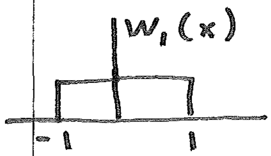
$$W_c(\vec{n}) = w(\|\vec{n}\|) \quad ; \|\vec{n}\| = \sqrt{n_1^2 + \ldots + n_M^2}$$

3. Rotated spectrum

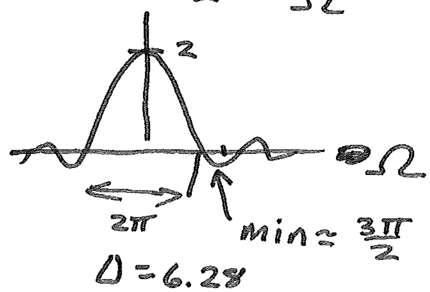
$$\overline{W}_s(\vec{\omega}) = \overline{W}(\|\vec{\omega}\|)$$

Window Comparison (normalize $\tau = 1$)
unalized

1. Rectangular



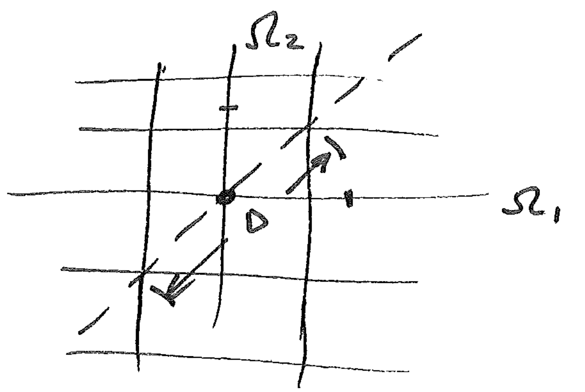
$2 \cdot \frac{\sin \Omega}{\Omega}$



$$2 \frac{\sin \frac{3\pi}{2}}{\frac{3\pi}{2}} = -0.4244$$

$\delta = 0.212 \approx 20\%$

Outer Product



in Ω_1 or Ω_2 direction, results are same as 1-D case.

$$W(\Omega_1, \Omega_2) = 4 \frac{\sin \Omega_1 \sin \Omega_2}{\Omega_1 \Omega_2}$$

Along $\Omega_1 = \Omega_2$

$$W(\Omega_1, \Omega_1) = \underbrace{4 \frac{\sin^2 \Omega_1}{\Omega_1^2}}$$

Bartlett window!

@ $\Omega_1 = \frac{3\pi}{2}$

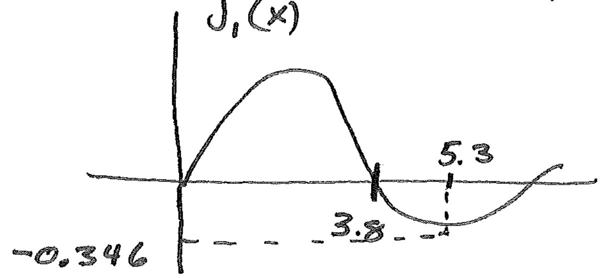
$$W(\Omega_1, \Omega_1) = 4 \frac{\sin^2 \frac{3\pi}{2}}{(\frac{3\pi}{2})^2} = 0.180$$

$\Rightarrow \delta = 0.04503$

But $\Delta = 2\pi\sqrt{2} = 8.886$

Rotated

$$W_2(\Omega_1, \Omega_2) = \frac{2\pi J_1 \sqrt{\Omega_1^2 + \Omega_2^2}}{\sqrt{\Omega_1^2 + \Omega_2^2}}$$



$W_2(0,0) = \pi$ (why?)

$\Rightarrow \Delta = 7.6$

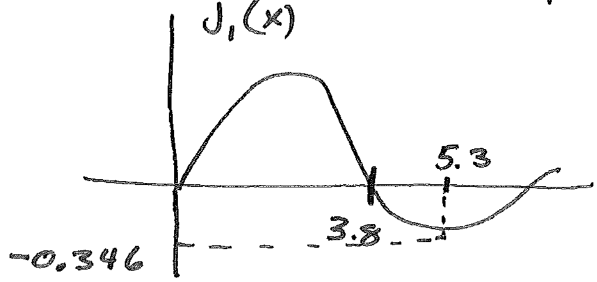
$$\delta = \frac{2\pi \cdot (-0.35) / 5.3}{\cancel{2}\pi} = 0.066$$

0.13

	Δ	δ
W_1	6.28	0.21
outer @45°	5.89	0.05
rotated	7.6	~~0.07~~ 0.13

Rotated

$$W_2(\Omega_1, \Omega_2) = \frac{2\pi J_1 \sqrt{\Omega_1^2 + \Omega_2^2}}{\sqrt{\Omega_1^2 + \Omega_2^2}}$$



$W_2(0,0) = \pi$ (why?)

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0.13

	Δ	δ
W_1	6.28	0.21
outer @45°	5.89	0.05
rotated	7.6	~~0.07~~ 0.13

"MULTIDIMENSIONAL PROJECTION WINDOWS"

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ABSTRACT

A one-dimensional window is chosen from the large catalog of those available primarily due to its leakage-resolution tradeoff (LRT). Is it possible to generalize a 1-D window to higher dimensions such that the window's 1-D properties are homogeneously preserved? If we require that the window be continuous and bounded the answer is usually no. Bounded (projection window) generalizations do exist for the Parzen and Tukey-Hanning windows. The resulting windows, however, are very close to that window obtained by simply rotating the 1-D window into two dimensions.

INTRODUCTION

When choosing from the large catalog of standard one-dimensional windows [1-2], one is largely motivated by the window's leakage-resolution tradeoff (LRT). Is it possible to generalize these windows to two and higher dimensions such that the 1-D window properties are preserved in each 1-D slice? If we require these multidimensional windows to be bounded and continuous, the answer is usually negative. In the two cases considered in this correspondence where bounded two dimensional generalizations do exist, the resulting windows are close to those obtained by the rotation generalization of 1-D windows [3].

A short review of the outer product and rotation of 1-D window generalization methods is given in the next section. In both cases, the LRT is altered in the transformation. In order to homogeneously maintain the 1-D window properties, the higher dimension window must be chosen so that its projection onto one dimension results in the 1-D window. Unfortunately, this requires unbounded generalizations in many cases of interest. The Parzen and Tukey-Hamming windows are the exceptions. For the discrete case, bounded projection windows can be formed such that desired LRT is preserved inhomogeneously at a number of angular orientations.

PRELIMINARIES

There are a wealth of one-dimensional windows with various leakage-resolution tradeoffs. A one-dimensional window, $w_1(t)$ has finite extent:

$$w_1(t) = w_1(t)\ \Pi\ (t/2\tau)$$

(where $\Pi(t) = 1$ for $|t| \leq 1/2$, and is zero elsewhere), is normalized with

$$w_1(0) = 1,$$

and is even function, i.e.,

$$w_1(t) = w_1(-t) \quad .$$

The spectrum of a window is defined by

$$W_1(\omega) = \int_{-\infty}^{\infty} w_1(t)\exp(-j\omega t)dt \quad .$$

The area of a window is

$$\begin{aligned} A &= \int_{-\infty}^{\infty} w_1(t)dt \\ &= W_1(0) \end{aligned}$$

The magnitude of a typical window spectrum is shown in Figure 1. For good resolution, the main lobe width, Δ, should be small, and for minimal spectral leakage, the normalized side lobe magnitude, δ, should also be small. Invariably, however, decreasing one of these parameters increases the other.

A two dimensional window $w_2(t_1, t_2)$, with spectrum

$$W_2(\omega_1, \omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w_2(t_1, t_2)\ \exp\ [-j(\omega_1 t_1+\omega_2 t_2)]dt_1 dt_2$$

is commonly generated from a 1-D counterpart by either the outer product or window rotation techniques [3]. The outer product window is

$$w_2^{op}(t_1, t_2) = w_1(t_1)\, w_1(t_2)$$

and the rotated window, initially suggested by Huang [4], is

$$w_2^{rw}(t_1, t_2) = w_1\left(\sqrt{t_1^2 + t_2^2}\right)$$

In either case, if w_1 is a "good" window, then so is w_2. For certain applications, (e.g. "good" filter design) such dimensional generalizations are acceptable. In other cases, such as spectral estimation, a small perturbation in window shape can significantly alter results [5]. Both the outer product and the rotated window significantly alter the LRT of the corresponding 1-D window.

To illustrate the effects of outer product and rotational dimensional generalization, we choose a boxcar window

$$w_1(t) = \Pi\,(t/2\tau)$$

It follows that:

$$W_1(\omega) = 2 \sin(\tau\omega)/\omega$$

for which

$$\Delta = 6.3 / \tau \;;\; \delta = 0.22 \qquad (1)$$

For the outer product window, in general:

$$W_2^{op}(\omega_1, \omega_2) = W_1(\omega_1)\, W_1(\omega_2)$$

The result is a window with an identical LRT as the 1-D window in the t_1 and t_2 directions. Indeed

$$W_2(\omega_1, 0) = A\, W_1(\omega_1)$$

However, in other directions, the LRT can be significantly altered. for example, in the (t_1, t_2) plane, the Δ parameter for the window resolution in the $\pm 45^\circ$ directions is $\sqrt{2}$ times that of the 0° and 90° directions. Consider, specifically, the boxcar window, for which

$$W_2^{op}(\omega_1, \omega_2) = 4 \sin(\tau\omega_1) \sin(\tau\omega_2) / (\omega_1\omega_2)$$

The 1-D slice of this window along the 45° diagonal is:

$$\begin{aligned} &W_2^{op}(\omega/\sqrt{2}, \omega/\sqrt{2}) \\ &= 4 \sin^2(\omega/\sqrt{2})/\omega^2 \end{aligned}$$

which is the spectrum of a Bartlett (triangular) window. The parameters of this window with respect to those in (1) are

$$\Delta_{45^\circ} = \sqrt{2}\Delta \simeq 8.9/\tau$$

and

$$\delta_{45^\circ} = 0.047 \simeq (0.22)^2 = \delta^2$$

Clearly, the LRT is significantly altered.

For the rotated window, the window spectrum can be written as

$$\begin{aligned} W_2^{rw}(\omega_1, \omega_2) &= W_2(\rho) \\ &= \int_0^\infty r\, W_1(r) J_0(r\rho)\, dr \qquad (2) \end{aligned}$$

where

$$\rho = \sqrt{\omega_1^2 + \omega_2^2}$$

and

$$r = \sqrt{t_1^2 + t_2^2}$$

Equation (2) is the familiar Hankel transform [6] which results from Fourier transforming a circularly symmetric 2-D function. Although the rotation window does not have the directional inhomogeneity of the outer product window, the LRT of the original window is also significatnly altered. Consider the rotated boxcar window with spectrum

$$W_2^{rw}(\rho) = 2\pi\tau J_1(\tau\rho)/\rho$$

Here

$$\Delta_{rw} \simeq 7.7/\tau = 1.2\Delta$$

and

$$\delta_{rw} = 0.13 \simeq 0.59\,\delta$$

THE PROJECTION OR ROTATED SPECTRUM WINDOW

The 2-D window, $w_2^p(r)$, that preserves the LRT of its corresponding 1-D window in all directions will be referred to as the projection or rotated spectrum window. The window can be thought of in one of two equivalent ways:

1. Projection

With reference to Fig. 2, $w_2^p(r)$ is the window whose projection is the 1-D design window:

$$w_1(t_1) = \int_{-\infty}^{\infty} w_2{}^p(r)\, dt_2 \tag{3}$$

By straightforward manipulation, w_1 is recognized as the Abel transform of w_2^p :

$$w_1(t_1) = 2 \int_{t_1}^{\infty} \frac{r\, w_2^p\,(r)dr}{\sqrt{r^2 - t_1{}^2}}$$

Thus, the 2-D window can be obtained from an inverse Abel transform [6]:

$$w_2^p\,(r) = \frac{1}{\pi} \int_r^{\infty} \sqrt{t_1{}^2 - r^2}\; \frac{d}{dt_1} \left[\frac{w_1'\,(t_1)}{t_1}\right] dt_1$$

where the prime denotes differentiation. Since $w_1(t_1)$ is zero for $|t_1| > \tau$, an equivalent expression is [6]:

$$w_2^p\,(r) = \frac{1}{\pi} \int_r^{\tau} \sqrt{t_1{}^2 - r^2}\; \frac{d}{dt_1} \left[\frac{w_1'\,(t_1)}{t_1}\right] dt_1$$

$$- \frac{w_1'\,(\tau)}{\pi\tau} \sqrt{\tau^2 - r^2}; \quad |r| \le \tau \tag{4}$$

2. Rotated Spectrum

The spectrum of the projection window is the rotation of the spectrum of the 1-D window. That is

$$W_2^p(\rho) = W_1(\rho)$$

The window can thus be obtained by an inverse Hankle transform:

$$w_2^p(r) = \int_0^\infty \rho \, W_1(\rho) \; J_o(r\rho)\, d\rho$$

Through this definition of the projection window, one can clearly see that the LRT of the original window is preserved in the 2-D generalization in all directions.

The equivalence of this and the projection window follows immediately from the continuous version of the projection - slice theorem [3] or, for even functions, from the equality of an Abel transform to Fourier Transform followed by an inverse Hankel transform [6].

EXAMPLES

1. The Parzen Window is obtained by convolving two identical (Barlett type) triangular windows and normalizing. The result is [7]:

$$w_1(t_1) = \begin{cases} 1 - 6\left(\frac{t_1}{\tau}\right)^2 + 6\left|\frac{t_1}{\tau}\right|^3 & ; |t_1| \leq \tau/2 \\ 2\left(1 - \left|\frac{t_1}{\tau}\right|\right)^3 & ; \tau/2 \leq |t_1| \leq \tau \\ 0 & ; |t_1| \geq \tau \end{cases}$$

Recognizing again that $w_1'(\tau) = 0$, we obtain from (4) after some variable substitution:

$$\hat{w}_2(r) = w_2^p(r\tau)$$

$$= \begin{cases} \frac{9}{\pi}\left[\frac{b}{2} - r^2 \ell n \left(\frac{\frac{1}{2} - b}{r}\right)\right] \\ + \frac{6}{\pi}\left[\frac{9b}{4} - \frac{3}{2}a + c\,\ell n\left(\frac{1+a}{\frac{1}{2}+b}\right)\right]; & 0 \le r \le \frac{1}{2} \\ \frac{6}{\pi}\left[\frac{-3a}{2} + c\,\ell n\,\frac{1+a}{r}\right] & ; 1/2 \le r \le 1 \end{cases}$$

where

$$a = \sqrt{1 - r^2}$$

$$b = \sqrt{\frac{1}{4} - r^2}$$

$$c = 1 + \frac{r^2}{2}$$

Plots of $\hat{w}_2(r)/w_2(0)$ and $w_1(t_1)$ (for $\tau = 1$) are shown in Fig. 3 using dashed and solid lines respectively. The difference between the two plots is nearly indistinguishable. Thus, the projection and rotation windows for the Parzen window are nearly identical.

2. **<u>The Tukey - Hanning Window</u>** is defined as

$$w_1\ (t) = \frac{1}{2}\ [1 + \cos(\frac{\pi t}{\tau})\ ]\ \Pi\ (t/2\tau)$$

Recognizing that $w_1'(\tau) = 0$, we can evaluate the resulting integral in (4) to obtain $w_2^p(r)$. Normalizing gives

$$\hat{w}_2(r) = w_2^p\ (r\tau)\ /\ \tau$$

$$= \frac{1}{2} \int_r^1 (\xi - r^2)^{\frac{1}{2}} \quad \frac{\pi\xi \cos(\pi\xi) - \sin(\pi\xi)}{\xi^2}\ d\xi$$

The integral can be easily evaluated numerically. Plots of $\hat{w}_2(r)/w_2(0)$ and $w_1(t_1)$ are shown in Fig. 4. The projection and rotation windows are again very similar.

BOUNDEDNESS OF THE PROJECTION WINDOW

A problem with certain continuous projection windows is their unboundedness. For example, the projection window corresponding to the boxcar window is

$$w_2\ (r) = \frac{1}{\pi(\tau^2 - r^2)^{1/2}}\ \Pi\ (r/2\tau)$$

This result is unbounded around the ring $r = \tau$. Similarly, for the Bartlett (triangular) window we obtain

$$w_2\ (r) = \frac{1}{\pi\tau}\ \cosh^{-1}(\tau/r)\ \Pi\ (r/2\tau)$$

This result is unbounded at the origin. Sufficient conditions for $w_2^p(r)$ to be bounded are :

$$\frac{d}{dt}\left[\frac{w_1'(t)}{t}\right] < \infty \tag{5}$$

and

$$\frac{dw_1(\tau)}{dt} < \infty \tag{6}$$

These conditions follow immediately upon inspection of (4). Equation (5), for example, is violated by the Bartlett window. Equation (6) excludes all 1-D windows that are discontinuous at $t = \tau$ (e.g., Hamming and Kaiser). The necessity of this can be seen in Figure 2. As in the vertical slice of $w_2^p(r)$ approaches $t_1 = \tau$ from the left, the circular support requires diminishingly smaller intervals of integration. The value of $w_1(\tau-)$ is determined by integration over an epsilon interval. Thus, in order for $w_1(\tau-)$ to be nonzero, $w_2^p(\tau-)$ must be infinite.

For digital signal processing, the boundedness of the projection window need not be a problem. Here, the 2-D window is set up in some given periodic grid (e.g. rectangular or hexogonal). The values in the window are chosen such that their projections [3] are the desired 1-D windows. A number of projection directions can be used. The result is a set of algebraic equations that can be solved to determine the values of the 2-D window. A second technique is to form a 2-D inverse FFT on the sampled windows's rotated spectrum. Some preliminary work in such digital extensions has been done by Wu [8].

EXTENSION TO HIGHER DIMENSIONS

For an N dimensional projection window, we wish to find $w_N^p(r_N)$ such that

$$w_1(r_1) = \int_{t_2}\int_{t_3}\cdots\int_{t_n} w_N^p(r_N)\, dt_N \ldots dt_3 dt_2 \quad , \tag{7}$$

where $w_1(r_1)$ is a specified 1-D window and

$$r_N^2 = \sum_{k=1}^{N} t_k^2$$

The integration in equation (7) can be done in stages, the Nth of which is

$$\begin{aligned} w_{N-1}(r_{N-1}) &= \int_{t_N} w_N(r_N)dt_N \\ &= \int_{t_N} w_N\left(\sqrt{r_{N-1}^2 + t_N^2}\right)dt_N \end{aligned}$$

Comparing with (3), we conclude that $w_{N-1}(r_{N-1})$ is the Abel transform of $w_N(r_N)$. Thus to generate $w_N(r_N)$, we simply need to perform N - 1 inverse Abel transforms on $w_1(t_1)$.

A pedagogical N = 5 closed form example, taken directly from an Abel transform table [6], is

$$w_1(r_1) = [1 - (\frac{r_1}{\tau})^2]\,\Pi(r_1/2\tau)$$

$$w_2(r_2) = \frac{2}{\pi\tau^2}(\tau^2 - r_2)^{1/2}\,\Pi(r_2/2\tau)$$

$$w_3(r_3) = \frac{1}{\pi\tau^2}\,\Pi(r_3/2\tau)$$

$$w_4(r_4) = \frac{1}{(\pi\tau)^2(\tau^2 - r_4^2)^{1/2}}\,\Pi(r_4/\tau)$$

$$w_5(r_5) = \frac{2}{\pi^2\tau}\,\delta(r_5 - \tau)$$

where δ is the unit impulse function.

An alternate approach to multidimensional projection windows follows from the property that the inverse Hankel transform of a Fourier transform is equivalent to an Abel transform. Thus, the N - 1 inverse Abel transform can be performed in the Fourier domain. Bracewell [6] has shown that these operations can be condensed into the single transform:

$$w_N\,(r_N) = \frac{N}{(2\pi r_N)^{N/2}} \int_0^{\infty} W_1(\omega)\; J_{N/2-1}\,(\omega r_N)\;\omega^{N/2}\,d\omega$$

where $J_{N/2-1}$ is the Bessel function of order N/2 -1.

CONCLUSIONS

The projection window preserves the leakage-resolution tradeoff (LRT) of the 1-D window from which it is designed. This is not in general true for the outer product and rotation window generalizations. The Parzen and Tukey-Hanning windows were shown to have straightforward two dimensional projection window equivalents. Many other commonly used windows, however, were shown to have unbounded projection. Further work in the digital equivalent of the dimensional generalization is in order. Here, boundedness need not be an issue.

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Figure Captions

Fig. 1: The normalized spectrum of a typical 1-D window, $|W_1(\epsilon)|/A$. The values of Δ and δ parameterize the window's resolution and leakage respectively.

Fig. 2: Illustration of the mechanics of forming a 1-D projection, $W_1(t_1)$, from a 2-D cicularly symmetric function $\hat{w}_2(r)$, ($r^2 = t_1^2 + t_2^2$). If $w_1(t_1)$ is the projection of $w_2(r)$, then $w_2(r)$ homogeneously preserves the LRT of its 1-D counterpart.

Fig. 3: Plots of the Parzen window, (dashed line) and its corresponding projection window, (solid line).

Fig. 4: Plots of the Tukey-Hamming window, and its corresponding projection window, (solid line).

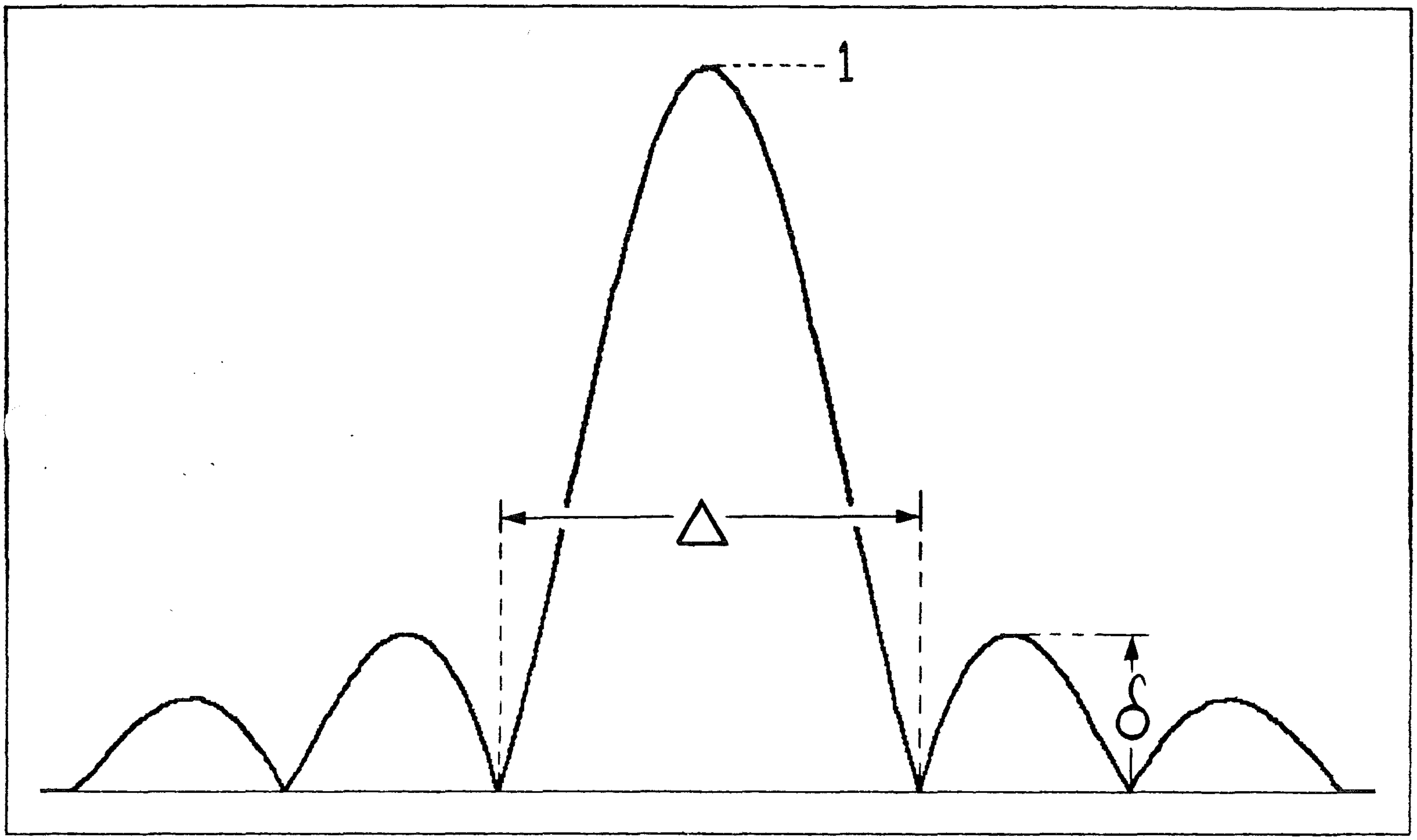


FIG 1

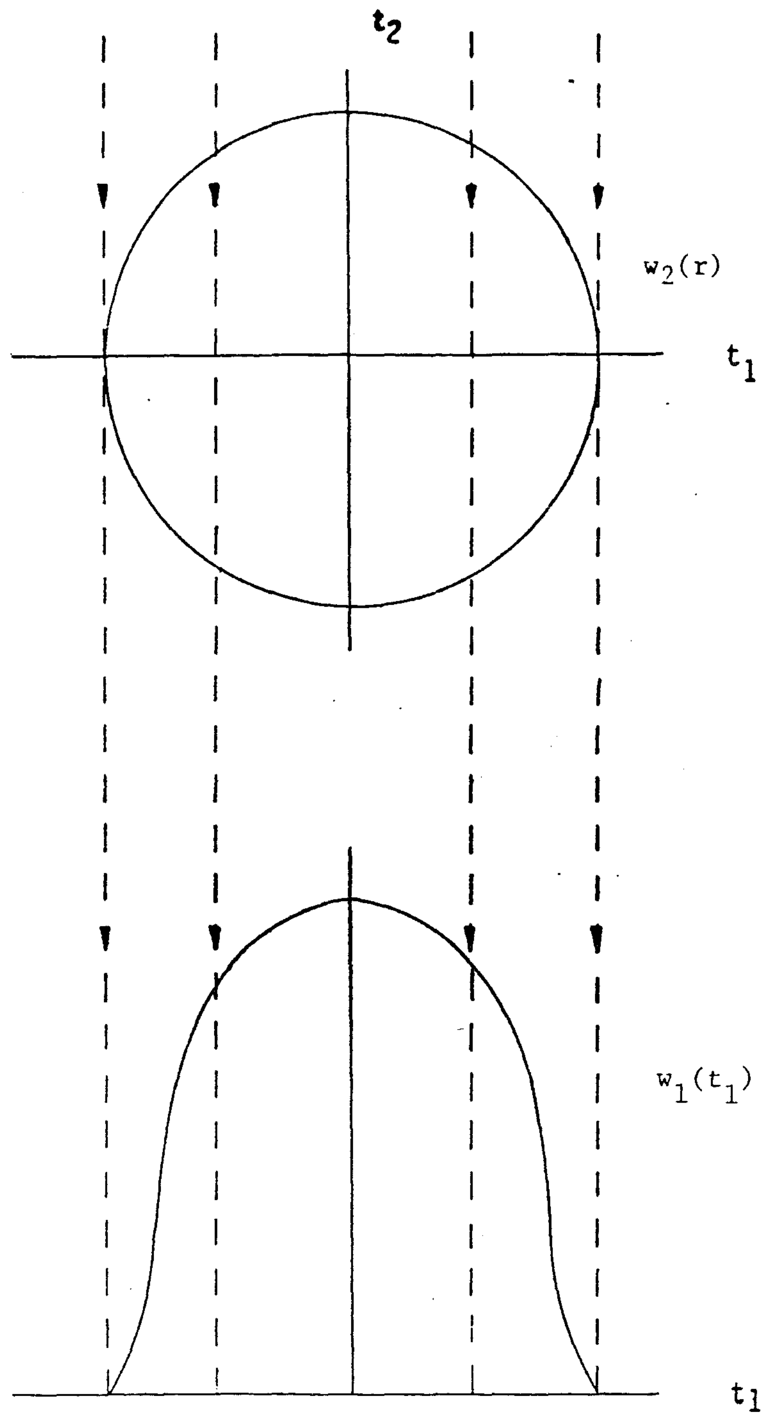


FIG 2

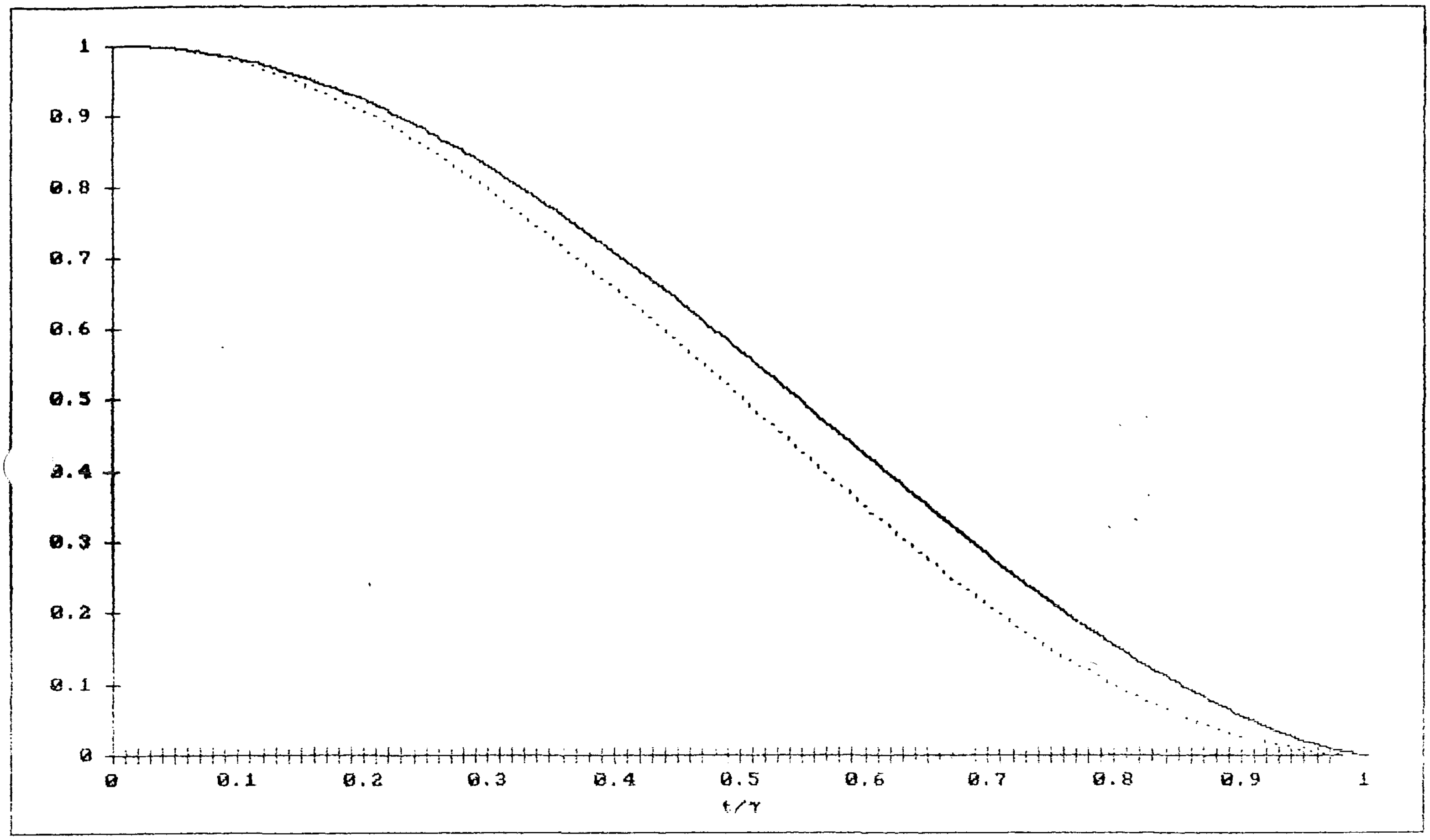


Fig 4

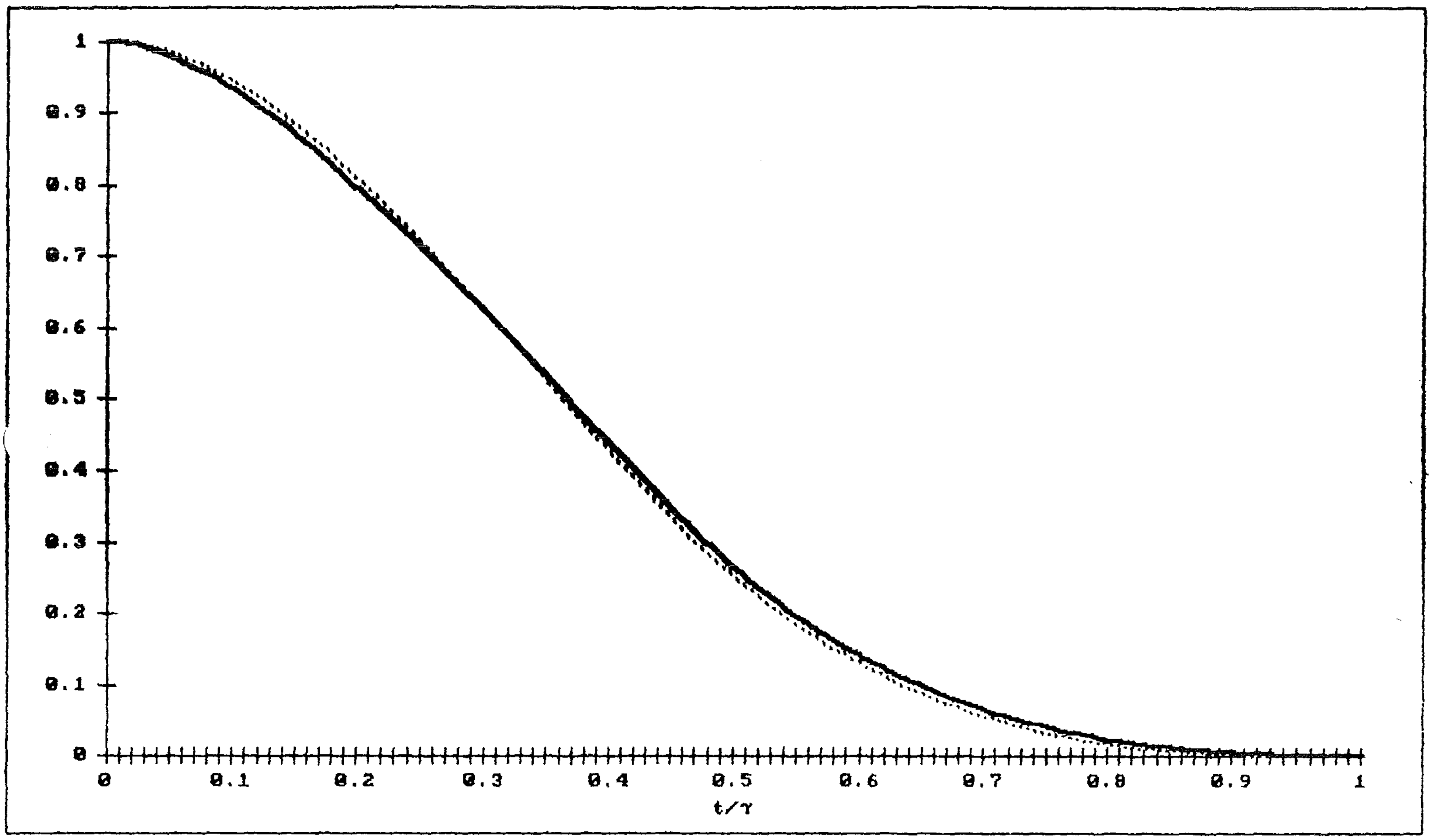
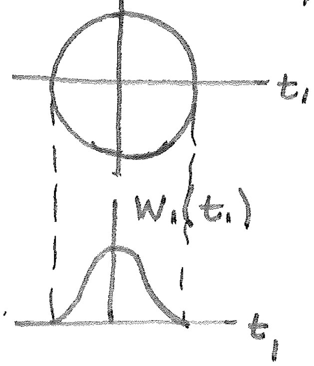


FIG 3

Rotated spectrum: (in - 2-D)

$$W_2(\Omega_1,\Omega_2) = W_1\left(\sqrt{\Omega_1^2+\Omega_2^2}\right)$$

$t_2 \quad w_2(t_1 t_2) = w_2(r)$



$$= 2\int_{r=t_1}^{\infty} w_2(r)\frac{r\,dr}{\sqrt{r^2-t_1^2}} \quad \left(\text{set } t_2=\sqrt{r^2-t_1^2}\right)$$

Theorem:

$$w_1(t_1) = \int_{-\infty}^{\infty} w_2(r)\,dt_2 \quad \Leftarrow \text{Abel transform}$$

$\cancel{W_2 t.}$

$$W_1(\Omega_1) = \int_{-\infty}^{\infty} \cancel{\frac{1}{2\pi}}\left[\int_{-\infty}^{\infty} w_2(r)\,dt_2\right] e^{-j\Omega_1 t_1}\,dt_1$$

$$W_2(\Omega_1,\Omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w_2(r)\, e^{-j(\Omega_1 t_1+\Omega_2 t_2)}\,dt_1\,dt_2$$

Thus: $W_1(\Omega_1) = W_2(\Omega_1 0)$

(elaborate)

Thus,

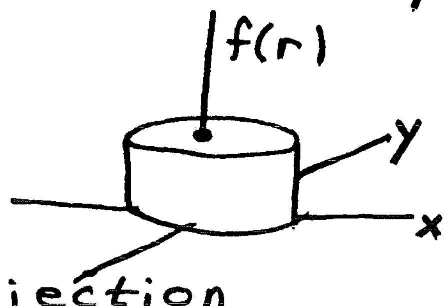
$w_1(t_1)$ is Abel transform of $w_2(r)$

$\therefore$ $w_2(r)$ is inverse Abel transform of $w(t_1)$.

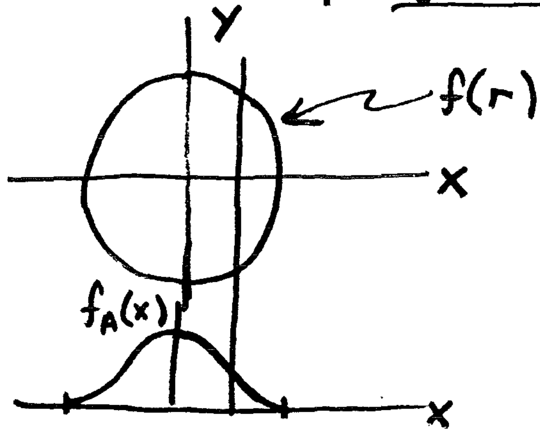
★ Abel Transform

Consider two dimensional radially symmetric function, $f(r)$, $r = \sqrt{x^2 + y^2}$

ex:



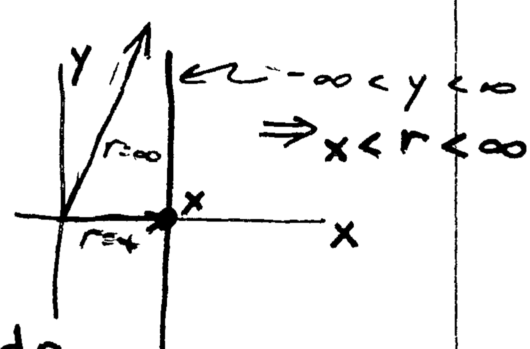
Look at projection



$$f_A(x) = \int_{-\infty}^{\infty} f(r)\,dy = 2\int_0^{\infty} f(r)\,dy$$

$$y = \sqrt{r^2 - x^2}$$

$$dy = \frac{r}{\sqrt{r^2 - x^2}}\,dr$$



Thus:

$$f_A(x) = 2\int_{r=x}^{\infty} \frac{r\,f(r)\,dr}{\sqrt{r^2 - x^2}} \Leftarrow \text{ABEL TRANSFORM}$$

IMPORTANT $\Longrightarrow$ MUST INVERT

$$u = \frac{f_A'(x)}{x} \qquad dV = \frac{x}{\sqrt{x^2 - r^2}}$$

$$du = \frac{d}{dx}\frac{f_A'(x)}{x} \qquad V = \sqrt{x^2 - r^2}$$

$$\Rightarrow f(r) = +\frac{1}{\pi}\int_r^{\infty} \sqrt{x^2 - r^2}\left[\frac{d}{dx}\frac{f_A'(x)}{x}\right] dx$$

$$+ \frac{1}{\pi} \cancel{\frac{f_A'(x)}{x} \sqrt{x^2 - r^2}} \Big|_{x=r}^{\infty} \quad 0$$

If f_A is not impulive @ $x = r_0$

If $f_A = 0 \ ; \ |r| > r_0$

$$f(r) = -\frac{1}{\pi}\int_r^{r_0} \sqrt{x^2 - r^2}\ \frac{d}{dx}\left[\frac{f_A'(x)}{x}\right] dx$$

$$- \frac{f_A'(r_0)}{\pi r_0}\sqrt{r_0^2 - r^2}$$

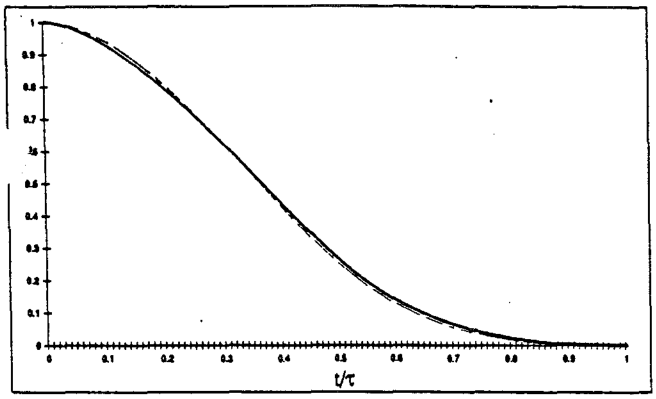


Fig. 3. Plots of the Parzen window (dashed line), and its corresponding projection window (solid line).

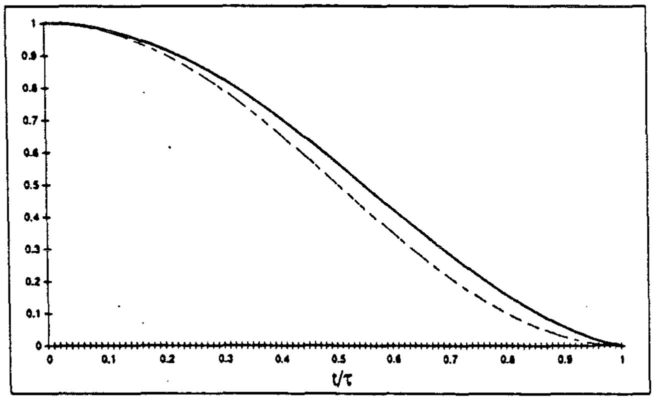


Fig. 4. Plots of the Tukey–Hanning window (dashed line), and its corresponding projection window (solid line).

where

$$a=(1-r^2)^{1/2}$$

$$b=\left(\frac{1}{4}-r^2\right)^{1/2}$$

$$c=1+\frac{r^2}{2}.$$

Plots of $\hat{w}_2(r)/w_2(0)$ and $w_1(t_1)$ (for $\tau=1$) are shown in Fig. 3 using dashed and solid lines, respectively. The difference between the two plots is nearly indistinguishable. Thus the projection and rotation windows for the Parzen window are nearly identical.

2) The *Tukey–Hanning Window* is defined as

$$w_1(t)=\frac{1}{2}\left(1+\cos\left(\frac{\pi t}{\tau}\right)\right)\Pi(t/2\tau).$$

Recognizing that $w_1'(\tau)=0$, we can evaluate the resulting integral in (4) to obtain $w_2^p(r)$. Normalizing gives

$$\hat{w}_2(r)=w_2^p(r\tau)/\tau$$
$$=\frac{1}{2}\int_r^1(\xi^2-r^2)^{1/2}\frac{\pi\xi\cos(\pi\xi)-\sin(\pi\xi)}{\xi^2}\,d\xi.$$

The integral can be easily evaluated numerically. Plots of $\hat{w}_2(r)/w_2(0)$ and $w_1(t_1)$ are shown in Fig. 4. The projection and rotation windows are again very similar.

Boundedness of the Projection Window

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This result is unbounded around the ring $r=\tau$. Similarly, for the Bartlett (triangular) window, we obtain

$$w_2(r)=\frac{1}{\pi\tau}\cosh^{-1}(\tau/r)\Pi(r/2\tau).$$

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$$\frac{d}{dt}\left(\frac{w_1'(t)}{t}\right)<\infty \tag{5}$$

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Extension to Higher Dimensions

For an N-D projection window, we wish to find $w_N(r_N)$ such that

$$w_1(r_1)=\int_{t_2}\int_{t_3}\cdots\int_{t_N}w_N(r_N)\,dt_N\cdots dt_3\,dt_2 \tag{7}$$

where $w_1(r_1)$ is a specified 1-D window and

$$r_N^2=\sum_{k=1}^N t_k^2.$$

The integration in equation (7) can be done in stages, the N-th of which is

$$w_{N-1}(r_{N-1})=\int_{t_N}w_N(r_N)\,dt$$
$$=\int_{t_N}w_N\left(\sqrt{r_{N-1}^2+t_N^2}\right)dt_N.$$

Comparing with (3), we conclude that $w_{N-1}(r_{N-1})$ is the Abel transform of $w_N(r_N)$. Thus to generate $w_N(r_N)$, we simply need to perform $N-1$ inverse Abel transforms on $w_1(t_1)$.

If $W_1(t) = W_1(t)\, p_\tau(t)$, then (Bracewell):

Inverse Abel transform:

$$W_2(r) = -\frac{1}{\pi}\int_r^\tau \sqrt{t^2-r^2}\;\frac{d}{dt}\left[\frac{\frac{dW_1(t)}{dt}}{t}\right]dt - \frac{\frac{dW_1(\tau)}{dt}}{\pi\tau}\sqrt{\tau^2-r^2}$$

Hamming Window: (Tukey

$$W_1(t) = \frac{1}{2}\left[1+\cos\frac{\pi t}{\tau}\right] p_\tau(t)$$

$$\frac{dW_1(t)}{dt} = \frac{1}{2}\,\frac{-\pi}{\tau}\sin\frac{\pi t}{\tau}\; p_\tau(t)$$

$$\frac{dW_1(\tau)}{dt} = 0$$

$$W_2(r) = -\frac{1}{\pi}\int_r^\tau \sqrt{t^2-r^2}\;\frac{d}{dt}\left(\frac{-\pi}{2\tau}\;\frac{\sin\frac{\pi t}{\tau}}{t}\right)dt$$

$$\cancel{=\frac{1}{2}}$$

$$= \frac{1}{2\tau}\int_r^\tau \sqrt{t^2-r^2}\;\frac{t\,\frac{\pi}{\tau}\cos\frac{\pi t}{\tau} - \sin\frac{\pi t}{\tau}}{t^2}\,dt$$

$$\cancel{W_2(r\tau) = \frac{1}{2\tau}\int_r^\tau}$$

$$\xi = t/r$$

$$\cancel{W_2(r) = \frac{1}{2\tau}\int_1^{\tau/r}\sqrt{(r\xi)^2-r^2}\;\frac{\frac{\pi r\xi}{\tau}\cos\frac{\pi r\xi}{\tau} - \sin\frac{\pi r\xi}{\tau}}{(r\xi)^2}\,r\,d\xi}$$

$$\cancel{W_2(r\tau) = \frac{1}{2\tau}\int_1^{r}\sqrt{\xi^2-1}\;\frac{\pi r\xi\cos\pi r\xi - \sin\pi r\xi}{r^2\tau^2\xi^2}\,d\xi}$$

etc

Must use digital integration. (Results) →

EVALUATION OF (45)

$$\frac{d}{dx}\,\frac{\sin\frac{\pi x}{\tau}}{x} = \frac{\frac{\pi x}{\tau}\cos\frac{\pi x}{\tau} - \sin\frac{\pi x}{\tau}}{x^2} \tag{55}$$

Thus (45) becomes:

$$W_2(r) = \frac{1}{2\tau}\int_r^{\tau} \sqrt{x^2 - r^2}\;\frac{\frac{\pi x}{\tau}\cos\frac{\pi x}{\tau} - \sin\frac{\pi x}{\tau}}{x^2}\,dx \tag{56}$$

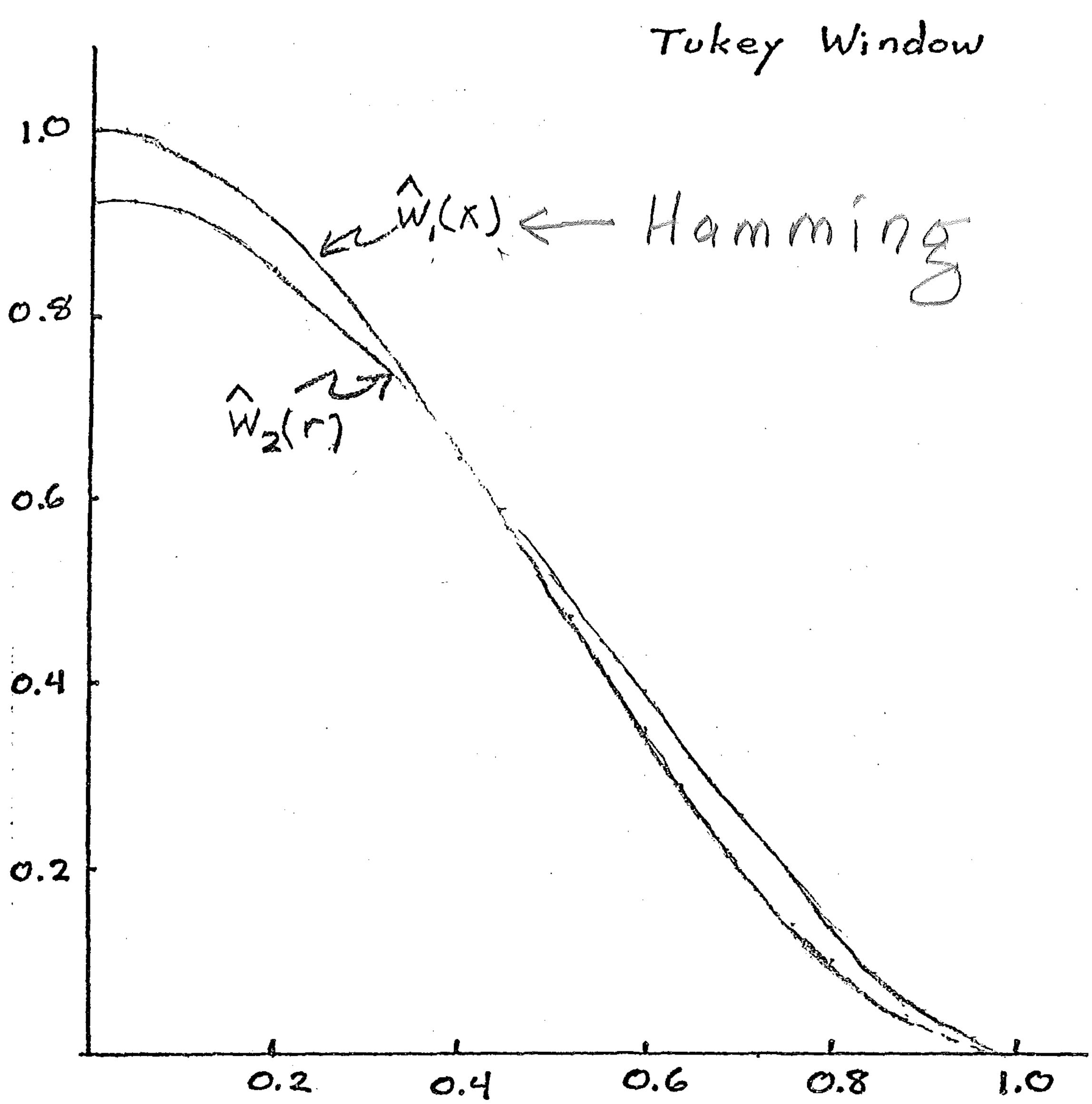
Set $\xi = x/\tau$

$$W_2(r) = \frac{1}{2\cancel{\tau}}\int_{r/\tau}^{1} \sqrt{(\tau\xi)^2 - r^2}\;\frac{\pi\xi\cos\pi\xi - \sin\pi\xi}{(\xi\cancel{\tau})^2}\,\cancel{\tau}\,d\xi \tag{57}$$

$$W_2(r\tau) = \frac{1}{2}\int_r^{1} \sqrt{(\tau\xi)^2 - (r\tau)^2}\;\frac{\pi\xi\cos\pi\xi - \sin\pi\xi}{\xi^2}\,d\xi \tag{58}$$

Or:

$$\frac{W_2(r\tau)}{\tau} = \frac{1}{2}\int_r^{1} \sqrt{\xi^2 - r^2}\;\frac{\pi\xi\cos\pi\xi - \sin\pi\xi}{\xi^2}\,d\xi \tag{59}$$

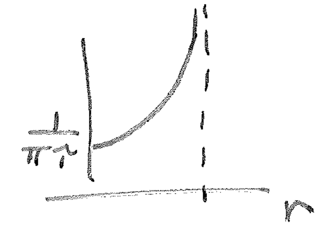


Tukey Window
Hamming
$\hat{w}_1(x)$
$\hat{W}_2(r)$
1.0
0.8
0.6
0.4
0.2
0.2
0.4
0.6
0.8
1.0

Problems:

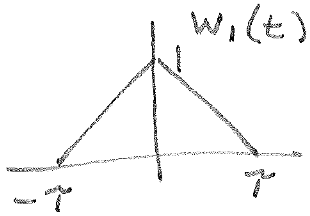
1. $w_1(\cancel{t}) = \rho_\tau(\cancel{t})$ (Boxcar)

$$w_2(r) = \frac{-1/\pi}{\sqrt{\tau^2 - r^2}}\, \rho_\tau(r)$$



2. Bartlett Window

$w_1(t) =$



$$w_2(r) = \frac{1}{\pi\tau} \cosh^{-1}\left(\frac{\tau}{r}\right) \rho_\tau(r)$$

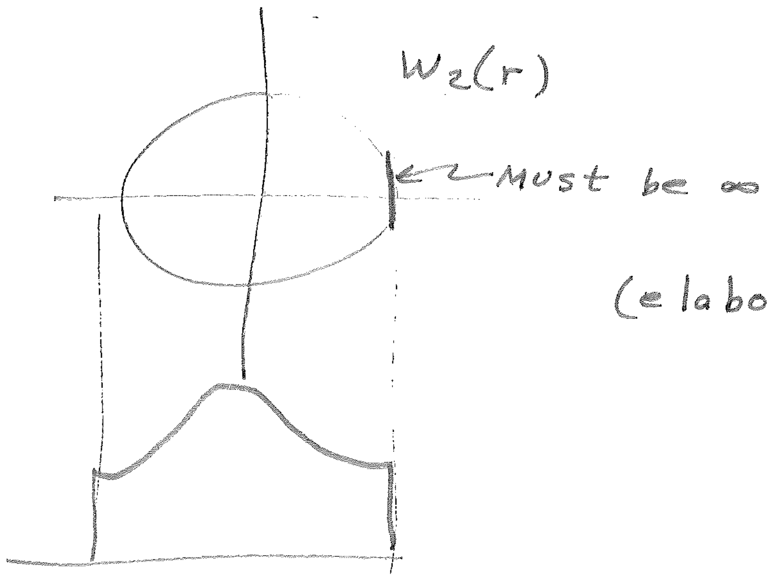
UNBOUNDED
@ r=0

Sufficient Conditions for $w_2(r)$ to be bounded

1. $w_1'(\tau) < \infty$
2. $\frac{d}{dx} \frac{dw_1(x)/dx}{x}$ is bounded

(see formula).

Will have singularity if $w_1(t)$ is discontinuous @ $t=\tau$ (elaborate)



(elaborate)

Kaiser, Hamming

MULTIDIMENSIONAL CASE

Define

$$r_n = \sqrt{\sum_{k=1}^{n} x_k^2} = \|\vec{x}\| \tag{60}$$

We wish to design an n dimensional window, $w_n(r_n)$, such that

$$w_1(r_1) = \int_{x_2}\int_{x_3}\cdots\int_{x_n} w_n(r_n)\, dx_2\, dx_3 \ldots dx_n \tag{61}$$

Note that this can be done in stages. ie

$$w_{n-1}(r_{n-1}) = \int_{x_n} w_n(r_n)\, dx_n \tag{62}$$

Proceeding:

$$w_{n-1}(r_{n-1}) = \int_{x_n=-\infty}^{\infty} w_n\left(\sqrt{r_{n-1}^2 + x_n^2}\right) dx_n \tag{63}$$

Make the substitution:

$$x_n = \sqrt{r_n^2 - r_{n-1}^2} \tag{64}$$

$$dx_n = \frac{r_n\, dr_n}{\sqrt{r_n^2 - r_{n-1}^2}} \tag{65}$$

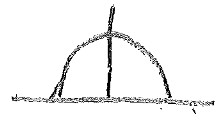
for $-\infty < x_n < \infty$, ~~$r_n > r_{n-1}$ and~~ $r_{n-1} \le r_n \le \infty$

$$w_{n-1}(r_{n-1}) = 2\int_{r_n=r_{n-1}}^{\infty} w_n(r_n)\, \frac{r_n\, dr_n}{\sqrt{r_n^2 - r_{n-1}^2}} \tag{66}$$

This is simply an Abel transform. Thus, for a given $w_1(x_1)$, we obtain $w_n(r_n)$ by performing n inverse Abel transforms.

Example: Parabola

$$w_1(r_1) = \frac{\tau^2 - r_1^2}{\tau^2} \, p_\tau(r_1) \tag{67}$$



$$w_2(r_2) = \frac{2}{\pi\tau^2} \sqrt{\tau^2 - r_2^2} \; p_\tau(r_2) \tag{68}$$

$$w_3(r_3) = \frac{1}{\pi\tau^2} \, p_\tau(r_3) \tag{69}$$

$$w_4(r_4) = \frac{1}{(\pi\tau)^2 \sqrt{\tau^2 - r_4^2}} \; p_\tau(r_4) \tag{70}$$

$$w_5(r_5) = \frac{2}{\pi^2\tau} \, \delta(r_5 - \tau) \tag{71}$$

The results are from Bracewell #2, p 264

Table 12.9 Some Abel transforms

$f(r)$		$f_A(x)$	
$\Pi(r/2a)$	Disk	$2(a^2 - x^2)^{\frac{1}{2}}\Pi(x/2a)$	Semiellipse
$(a^2 - r^2)^{-\frac{1}{2}}\Pi(r/2a)$		$\pi\Pi(x/2a)$	Rectangle
$(a^2 - r^2)^{\frac{1}{2}}\Pi(r/2a)$	Hemisphere	$\frac{1}{2}\pi(a^2 - x^2)\Pi(x/2a)$	Parabola
$(a^2 - r^2)\Pi(r/2a)$	Paraboloid	$\frac{4}{3}(a^2 - x^2)^{\frac{3}{2}}\Pi(x/2a)$	
$(a^2 - r^2)^{\frac{3}{2}}\Pi(r/2a)$		$(3\pi/8)(a^2 - x^2)^2\Pi(x/2a)$	
$a\Lambda(r/a)$	Cone	$[a(a^2 - x^2)^{\frac{1}{2}} - x^2 \cosh^{-1}(a/x)]\Pi(x/2a)$	
$\pi^{-1}\cosh^{-1}(a/r)\Pi(r/2a)$		$a\Lambda(x/a)$	Triangle
$\delta(r - a)$	Ring impulse	$2a(a^2 - x^2)^{-\frac{1}{2}}\Pi(x/2a)$	
$\exp(-r^2/2\sigma^2)$	Gaussian	$(2\pi)^{\frac{1}{2}}\sigma \exp(-x^2/2\sigma^2)$	Gaussian
$r^2 \exp(-r^2/2\sigma^2)$		$(2\pi)^{\frac{1}{2}}\sigma(x^2 + \sigma^2)\exp(-x^2/2\sigma^2)$	
$(r^2 - \sigma^2)\exp(-r^2/2\sigma^2)$		$(2\pi)^{\frac{1}{2}}\sigma x^2 \exp(-x^2/2\sigma^2)$	
$(a^2 + r^2)^{-1}$		$\pi(a^2 + x^2)^{-\frac{1}{2}}$	
$J_0(2\pi ar)$		$(\pi a)^{-1}\cos 2\pi ax$	
$2\pi\left[r^{-3}\int_0^r J_0(r)\,dr - r^{-2}J_0(r)\right] = M(r)$		$\operatorname{sinc}^2 x$	
$\delta(r)/\pi\lvert r\rvert$		$\delta(x)$	
$2a \operatorname{sinc} 2ar$		$J_0(2\pi ax)$	
$\frac{1}{2}r^{-1}J_1(2\pi ar)$		$\operatorname{sinc} 2ax$	

Since $\bar{K}$ is nowhere zero, the solution is unique (except for additive null functions).

Reverting to f and f_A, we may write the solutions as

$$f(r) = -\frac{1}{\pi}\int_r^\infty \frac{f'_A(x)\,dx}{(x^2 - r^2)^{\frac{1}{2}}} = +\frac{1}{\pi}\int_r^\infty (x^2 - r^2)^{\frac{1}{2}} \frac{d}{dx}\left[\frac{f'_A(x)}{x}\right] dx,$$

or, if the integral is zero beyond $x = r_0$, and allowing for the possibility that the integrand may behave impulsively at r_0, we have

$$f(r) = -\frac{1}{\pi}\int_r^{r_0} \frac{f'_A(x)\,dx}{(x^2 - r^2)^{\frac{1}{2}}} + \frac{f_A(r_0)}{\pi(r_0^2 - r^2)^{\frac{1}{2}}}$$
$$= -\frac{1}{\pi}\int_r^{r_0} (x^2 - r^2)\frac{d}{dx}\left[\frac{f'_A(x)}{x}\right] dx - \frac{f'_A(r_0)}{\pi r_0}(r_0^2 - r^2)^{\frac{1}{2}}.$$

Useful relations for checking Abel transforms are

$$\int_{-\infty}^{\infty} f_A(x)\,dx = 2\pi\int_0^\infty f(r)r\,dr$$

and

$$f_A(0) = 2\int_0^\infty f(r)\,dr.$$

Another property is that

$$K * K * F' = -\pi F;$$

that is, the operation $K\,*$ applied twice in succession annuls differentiation; then F_A is the half-order integral of F, and conversely, F is the half-order differential coefficient of F_A. To prove this, note that if $F_A = K * F$ implies that $F = -\pi^{-1}K * F'_A$, then it follows further that $F'_A = K * F'$; whence

$$K * K * F' = K * F'_A = -\pi F.$$

In Table 12.9 the first eight examples are to be taken as zero for r and x greater than a.

Numerical evaluation of Abel transforms is comparatively simple in view of the possibility of conversion to a convolution integral. One first makes the change of variable, then evaluates sums of products of $K(\rho)$ and $f(\xi - \rho)$ at discrete intervals of ρ. The values of K turn out to be the same, however fine an interval is chosen, save for a normalizing factor; consequently, a universal table of values (see Table 12.10) can be set up for permanent reference. The table shows coefficients for immediate use with values of F read off at $\rho = \frac{1}{2}, 1\frac{1}{2}, \ldots, 9\frac{1}{2}$, the scale of ρ being such that F becomes zero or negligible at $\rho = 10$. The table gives mean values of K over the intervals $0 - 1, 1 - 2, \ldots.$ Thus at $\rho = n + \frac{1}{2}$ the value is

$$\int_n^{n+1} K(-\rho)\,d\rho = 2(n + 1)^{\frac{1}{2}} - 2n^{\frac{1}{2}}.$$

Table 12.10 Coefficients for performing or inverting the Abel transformation

ρ	K	ρ	K	ρ	K	ρ	K
$\frac{1}{2}$	2.000	$5\frac{1}{2}$	0.427	$10\frac{1}{2}$	0.309	$15\frac{1}{2}$	0.254
$1\frac{1}{2}$	0.828	$6\frac{1}{2}$	0.393	$11\frac{1}{2}$	0.295	$16\frac{1}{2}$	0.246
$2\frac{1}{2}$	0.636	$7\frac{1}{2}$	0.364	$12\frac{1}{2}$	0.283	$17\frac{1}{2}$	0.239
$3\frac{1}{2}$	0.536	$8\frac{1}{2}$	0.343	$13\frac{1}{2}$	0.272	$18\frac{1}{2}$	0.233
$4\frac{1}{2}$	0.472	$9\frac{1}{2}$	0.325	$14\frac{1}{2}$	0.263	$19\frac{1}{2}$	0.226

$$F_L(p) = \int_{-\infty}^{\infty} f(t)e^{-pt}\,dt \qquad f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F_L(p)e^{pt}\,dp$$

$$F_M(s) = \int_0^{\infty} f(x)x^{s-1}\,dx \qquad f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F_M(s)x^{-s}\,ds$$

$$f(n) = \frac{1}{2\pi i}\int_{\Gamma} F(z)z^{n-1}\,dz \qquad F(z) = \int_{-\infty}^{\infty} \phi(t)z^{-t}\,dt = \sum_0^{\infty} f(n)z^{-n}$$

It is clear that the z transform is like the inverse Mellin transform except that t must assume real values whereas s may be complex, and conversely, x is real whereas z may be complex. The contour Γ on the z plane may be understood as follows. It must enclose the poles of the integrand. If the contour $c - i\infty$ to $c + i\infty$ for inverting the Laplace transformation is chosen to the right of all poles, then the circle into which it is transformed by the transformation $z = \exp(-p)$ will enclose all poles. In the common case where $c = 0$ is suitable (all poles of $F_L(p)$ in the left half-plane), the contour Γ becomes the circle $|z| = 1$.

The Abel transform

As soon as one goes beyond the one-dimensional applications of Fourier transforms and into optical-image formation, television-raster display, mapping by radar or passive detection, and so on, one encounters phenomena which invite the use of the Abel transform for their neatest treatment. These phenomena arise when circularly symmetrical distributions in two dimensions are projected in one dimension. A typical example is the electrical response of a television camera as it scans across a narrow line; another is the electrical response of a microdensitometer whose slit scans over a circularly symmetrical density distribution on a photographic plate.

Fractional-order derivatives are also closely connected with the Abel transform, which therefore also arises in fields, such as conduction of heat in solids or transmission of electrical signals through cables, where fractional-order derivatives are encountered.

The Abel transform $f_A(x)$ of the function $f(r)$ is commonly defined as

$$f_A(x) = 2\int_x^{\infty} \frac{f(r)r\,dr}{(r^2 - x^2)^{\frac{1}{2}}}.$$

The choice of the symbols x and r is suggested by the many applications in which they represent an abscissa and a radius, respectively, in the same plane.

The above formula may be written

$$f_A(x) = \int_0^{\infty} k(r,x)f(r)\,dr,$$

where

$$k(r,x) = \begin{cases} 2r(r^2 - x^2)^{-\frac{1}{2}} & r > x \\ 0 & r < x. \end{cases}$$

The kernel $k(r,x)$, regarded as a function of r in which x is a parameter, shifts to the right as x increases, and it also changes its form. A slight change of variable leads to a kernel which simply shifts without change of form. Thus putting $\xi = x^2$ and $\rho = r^2$, and letting $f_A(x) = F_A(x^2)$ and $f(r) = F(r^2)$, we have

$$F_A(\xi) = \int_0^{\infty} K(\xi - \rho)F(\rho)\,d\rho,$$

where

$$K(\xi) = \begin{cases} (-\xi)^{-\frac{1}{2}} & \xi < 0 \\ 0 & \xi \geqslant 0; \end{cases}$$

alternatively,

$$F_A(\xi) = \int_\rho^{\infty} \frac{F(\rho)\,d\rho}{(\rho - \xi)^{\frac{1}{2}}},$$

or again,

$$F_A = K * F.$$

When necessary, F_A will be referred to as the "modified Abel transform of F." Having reduced the formula to a convolution integral, we may take Fourier transforms and write

$$\bar{F}_A = \bar{K}\bar{F}.$$

Since

$$\bar{K}(s) = \frac{1}{(-2is)^{\frac{1}{2}}},$$

if follows that

$$\bar{F} = (-2is)^{\frac{1}{2}}\bar{F}_A = -\frac{1}{\pi}\frac{1}{(-2is)^{\frac{1}{2}}}\,i2\pi s\bar{F}_A$$

whence

$$F = -\frac{1}{\pi}K * F'_A;$$

that is,

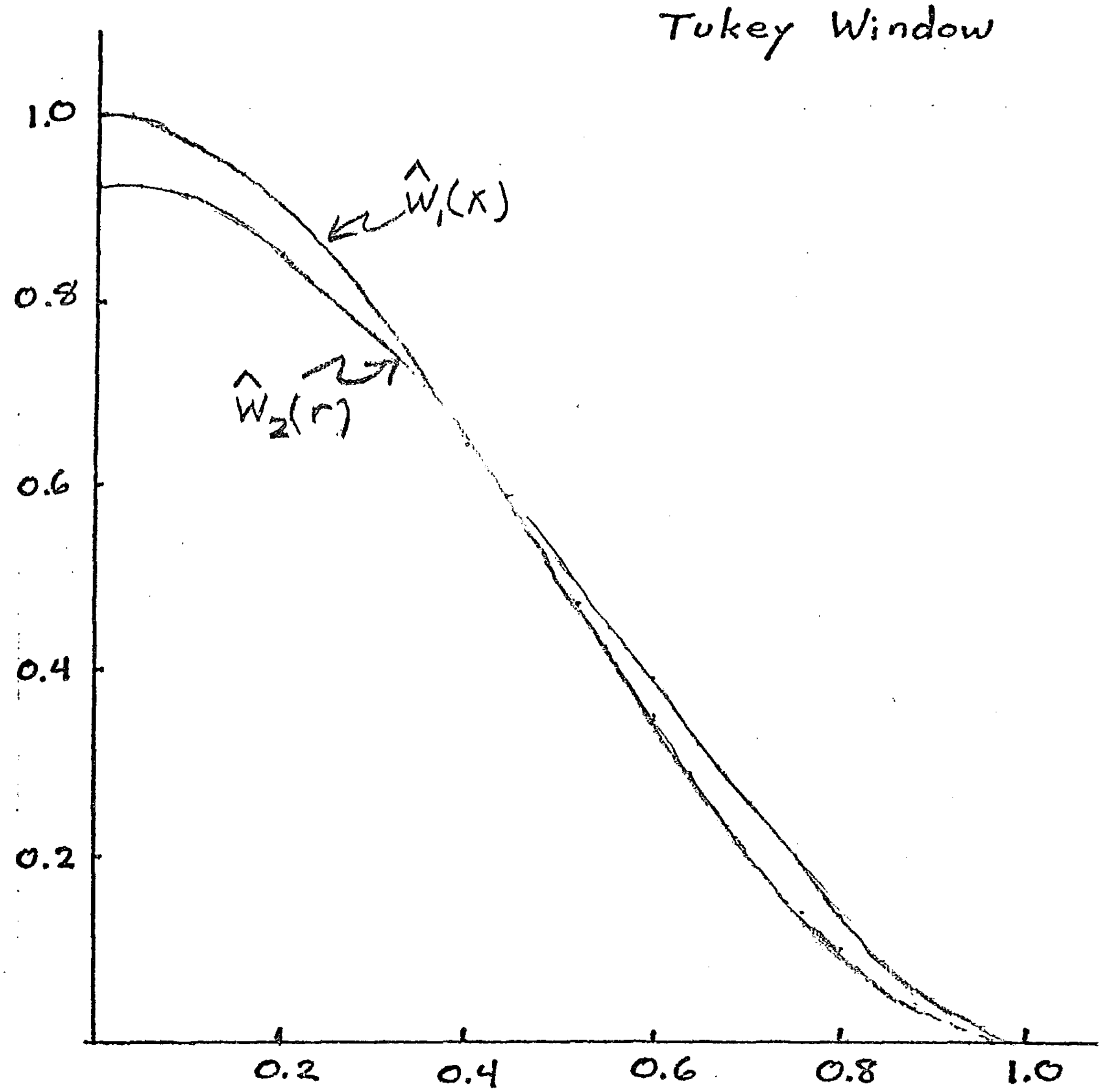
$$F(\rho) = -\frac{1}{\pi}\int_\rho^{\infty} \frac{F'_A(\xi)\,d\xi}{(\xi - \rho)^{\frac{1}{2}}},$$

The solution of the modified Abel integral equation enables F to be expressed in terms of the derivative of F_A. Integrating the solution by parts, or choosing different factors for the transform of F, we obtain a solution in terms of the second derivative of F_A:

$$F = \frac{2}{\pi}\mathcal{K} * F''_A,$$

where

$$\mathcal{K}(\xi) = \begin{cases} (-\xi)^{\frac{1}{2}} & \xi < 0 \\ 0 & \xi \geqslant 0. \end{cases}$$



Tukey Window
1.0
0.8
0.6
0.4
0.2
$\hat{w}_1(x)$
$\hat{w}_2(r)$
0.2
0.4
0.6
0.8
1.0

3.4. Optimal FIR filter design

$$E(\vec{\omega}) = H(\vec{\omega}) - I(\vec{\omega})$$

error FIR WANT

Minimize error in some sense, e.g. L_p

$$E_p = \left[\frac{1}{(2\pi)^M} \int_{\pi} |E(\vec{\omega})|^p d\vec{\omega} \right]^{\frac{1}{p}}$$

For FIR filter, let support be in R

$$H(\vec{\omega}) = \sum_{\vec{n} \in R} h[\vec{n}] e^{-j\vec{\omega}^T \vec{n}}$$

Thus

$$E(\vec{\omega}) = \sum_{\vec{n} \in R} h[\vec{n}] e^{-j\vec{\omega}^T \vec{n}} - I(\vec{\omega})$$

Aside

Reducing D.O.F.

D.O.F. = # of $h[\vec{n}]$'s you gotta know

In 1-D;

if $h[n]$ is even, can reduce DOF by $\sim \frac{1}{2}$

$$H(\omega) = h[0] + 2\sum_{n=1}^{N} h[n] \cos(\omega n)$$

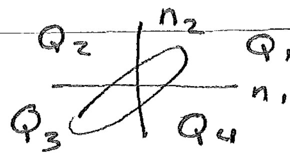
In 2-D

$$H(\omega_1 \omega_2) = \sum_{n_1} \sum_{n_2 \in R} h[n_1 n_2] e^{-j(\omega_1 n_1 + \omega_2 n_2)}$$

If $h[n_1 n_2] = h[-n_1 - n_2]$

$$H(\omega_1 \omega_2) = \sum_{n_1 n_2 \in R} \sum h[n_1 n_2] \cos(\omega_1 n_1 + \omega_2 n_2)$$

$$= \cancel{h(0,0) + 2\sum_{\substack{n_1 > 0 \\ n_2}}}$$



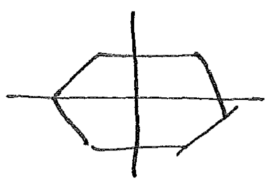
$$H(\omega_1,\omega_2) = h(0,0) + \Big(\sum_{Q_1 Q_2} + \sum_{Q_3 Q_4}\Big) h[n_1,n_2] \times \cos(\omega_1 n_1 + \omega_2 n_2)$$

In $Q_3\ Q_4$, set $n_1 = -n_1$, $n_2 - n_2$

$$\Rightarrow H(\omega_1,\omega_2) = h(0,0) + 2 \underbrace{\sum_{Q_1 Q_2}}_{R'} h[n_1,n_2] \cos(\omega_1 n_1 + \omega_2 n_2)$$

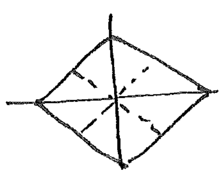
ABOUT $\frac{1}{2}$ the D.O.F.

If : $h[n_1,n_2] = h[-n_1,-n_2] = h[n_1,-n_2] = h[-n_1, n_2]$



Reduce D.O.F. by 4

If: $h[n_1,n_2] = h[\pm n_1, \pm n_2] = h[\pm n_2, \pm n_1]$



Reduce D.O.F. by 8

For zero phase $h[n_1, n_2] = h[-n_1, -n_2]$

~~Further~~ Simplification into D.O.F.

$$H(\vec{\omega}) = \sum_{i=1}^{F = D.O.F.} a(i)\, \phi_i(\vec{\omega}) \leftarrow \text{Basis Function}$$

For this case:

$$\phi_i(\omega_1, \omega_2) = \begin{cases} 2\cos(\omega_1 n_1 + \omega_2 n_2) & ;\ (n_1, n_2) \neq (0,0) \\ 1 & (n_1, n_2) = (0,0) \end{cases}$$

$$a(i) = h[n_1, n_2]$$

~~Easy to~~

Easy to alter

If $a(i) = a(j)$

Replace $a(i)\phi_i$ by $[a(i) + a(j)]\phi_i$,

Delete j term. DOF $\downarrow$ by 1

Specify $a(i) = k \Rightarrow$ reduce D.O.F.

Optimal FIR Filter Design

$$\underset{\text{ERROR}}{E(\vec{\omega})} = \underset{\text{FIR}}{H(\vec{\omega})} - \underset{\text{WANT}}{I(\vec{\omega})}$$

Minimize Error in some sense, e.g. LP norm:

$$E_p = \left[\frac{1}{(2\pi)^M}\int_{\mathcal{F}} |E(\vec{\omega})|^p d\vec{\omega}\right]^{1/p}$$

3.4.1. OR, LEAST SQUARES DESIGN

$$E_2 = \frac{1}{(2\pi)^M}\int_{\mathcal{F}} |E(\vec{\omega})|^2 d\vec{\omega}$$

~~R~~ $\cancel{= \frac{1}{(2\pi)^M}\int_{\mathcal{F}}}$

Parseval's Theorem

$$= \sum_{\vec{n}} |e[\vec{n}]|^2 d\vec{n}$$

$$= \sum_{\vec{n}} |h[\vec{n}] - i[\vec{n}]|^2$$

$$= \Big(\underbrace{\sum_{\vec{n}\in R}}_{\text{FIR}} + \sum_{\vec{n}\notin R}\Big) |h[\vec{n}] - i[\vec{n}]|^2$$

But $h[\vec{n}] = 0$ for $\vec{n} \notin R$

$$E_2 = \sum_{\vec{n}\in R} |h[\vec{n}] - i[\vec{n}]|^2 + \sum_{\vec{n}\notin R} |i[\vec{n}]|^2$$

$i[n_1, n_2]$ is fixed.

Minimize by choosing $h[\vec{n}] = \begin{cases} i[\vec{n}]; & \vec{n}\in R \\ 0; & \vec{n}\notin R \end{cases}$

Same as flat window!

(elaborate)

$\cancel{h[n_1 n_2] =}$ $h[\cancel{\vec{n}}]$

For zero phase

$$H(\omega_1, \omega_2) = h[0,0] + 2 \sum_{Q_1 Q_2} h[n_1 n_2] \cos(\omega_1 n_1 + \omega_2 n_2)$$

If $h[n_1 n_2] = h[-n_1 n_2]$

$$H(\omega_1, \omega_2) = h[0,0] + 2 \sum_{Q_1} h[n_1 n_2] \left(\cos(\omega_1 n_1 + \omega_2 n_2) + \cos(\omega_1 n_1 - \omega_2 n_2)\right)$$

$$= \sum_{i=1}^{F} a[i] \phi_i(\omega_1 \omega_2)$$

$$a[i] = \begin{cases} 2h[n_1 n_2] & ; (n_1, n_2) \neq (0,0) \\ h[0,0] & ; (n_1 n_2) = (0,0) \end{cases}$$

$$\phi_i = \cos(\omega_1 n_1 + \omega_2 n_2) + \cos(\omega_1 n_1 - \omega_2 n_2)$$

Alternate formulation

$$\hat{H}(\vec{\omega}) = \sum_{i=1}^{F} a[i]\,\phi_i(\vec{\omega})$$

$$E_2 = \frac{1}{(2\pi)^M}\int_{\mathcal{F}} \left| H(\vec{\omega}) - I(\vec{\omega}) \right|^2 d\vec{\omega}$$

$$= \frac{1}{(2\pi)^M}\int_{\mathcal{F}} \left| \sum_{i=1}^{F} a[i]\,\phi_i(\vec{\omega}) - I(\vec{\omega}) \right|^2 d\vec{\omega}$$

Take $\frac{\delta E_2}{\delta a_k} = 0$; $k = 1, 2, \ldots F$

F eqs & F unknowns

$$E_2 = \frac{1}{(2\pi)^M}\int_{\mathcal{F}} \Big[I^2(\vec{\omega}) + \sum_{i=1}^{F}\sum_{\ell=1}^{F} a[i]\,a[\ell]\,\phi_i(\vec{\omega})\,\phi_j(\vec{\omega})$$
$$- 2 I(\vec{\omega}) \sum_{i=1}^{F} a[i]\,\phi_i(\omega) \Big] d\vec{\omega}$$

$\cancel{\frac{\delta E_2}{\delta a_k} = \frac{1}{(2\pi)^M}\int_{\mathcal{F}}}$

$a[i]\,a[\ell]$

$i \longrightarrow k$

$\ell \downarrow$ $a(k)a(1)$

$\vdots$

k $a(k)a(1)\cdots a^2(k) + \ldots a(k)a(F)$

$\vdots$

$a(k)a(F)$

Direct

$$\frac{\delta E_2}{\delta a_k} = \frac{1}{(2\pi)^M}\int_{\mathcal{F}} \Big[2\sum_{\substack{i=1 \\ i \neq k}}^{F} a[i]\,\phi_i(\vec{\omega})\,\phi_k(\vec{\omega})$$
$$+ 2a[k]\,\phi_k^2(\vec{\omega})$$
$$\cancel{-2I(\vec{\omega})\sum_{i=1}^{F} a} - 2I(\vec{\omega})\,\phi_k(\vec{\omega}) \Big] d\vec{\omega}$$

$$= \frac{2}{(2\pi)^M} \int_{\Pi} \left(\sum_{i=1}^{F} a[i]\, \phi_i(\vec{\omega}) - \cancel{2} I(\vec{\omega}) \cancel{a[k]} \right) \phi_k(\vec{\omega})\, d\vec{\omega}$$

$$= 0$$

Define $\quad \phi_{ik} = \frac{1}{(2\pi)^M} \int_{\Pi} \phi_i(\vec{\omega})\, \phi_k(\vec{\omega})\, d\vec{\omega}$

$$I_k = \frac{1}{(2\pi)^M} \int_{\Pi} I(\omega)\, \phi_k(\vec{\omega})\, d\vec{\omega}$$

$$\Rightarrow 2 \left[\sum_{i=1}^{F} a[i]\, \phi_{ik} - I_k \cancel{a[k]} \right] = 0$$

$$\sum_{i=1}^{F} a[i]\, \phi_{ik} = I_k \quad \Longleftarrow \begin{array}{l} F \text{ eqs} \\ F \text{ unknowns} \end{array}$$

If ϕ_i's orthogonal

$$\phi_{ik} = 0 \quad ; i \neq k$$

$$\Rightarrow a[i] = I_i / \phi_{ii}$$

Problem with E_2 : ripples (Gibbs).

Alternate Technique. Require equality at pts:

$$H(\vec{\omega}) = I(\vec{\omega}) \quad \text{for } \vec{\omega} = (\vec{\omega}_1, \vec{\omega}_2, \ldots, \vec{\omega}_F)$$

$$I(\vec{\omega}_k) = H(\vec{\omega}_k) = \sum_{\vec{n} \in R} \underbrace{h[\vec{n}]}_{F\ \#\text{'s}} e^{-j\vec{\omega}_k^T \vec{n}} \; ; k = 1, \ldots, F$$

F eqs & F unknowns

If we choose $\quad \vec{\omega}_k^{\mathbf{T}} = \vec{k}^{\mathbf{T}} 2\pi \underline{N}^{-1}$ $\qquad \underline{N} = \begin{bmatrix} N_1 & & \\ & \ddots & \\ & & N_M \end{bmatrix}$

$$\vec{\omega}_k = 2\pi \underline{N}^{-1^T} \vec{k}$$

and $\quad R = N_1 \times N_2 \times \ldots \times N_M$ hypercube, then

$$I(\vec{\omega}_k) = H(\vec{k}) = \text{DFT of } h[\vec{n}]$$

$\Rightarrow h[\vec{n}]$ is inverse DFT of $I(\vec{\omega}_k)$

$$= I(2\pi \underline{N}^{-1} \vec{k}):$$

p. 65
2.20
& 2.21

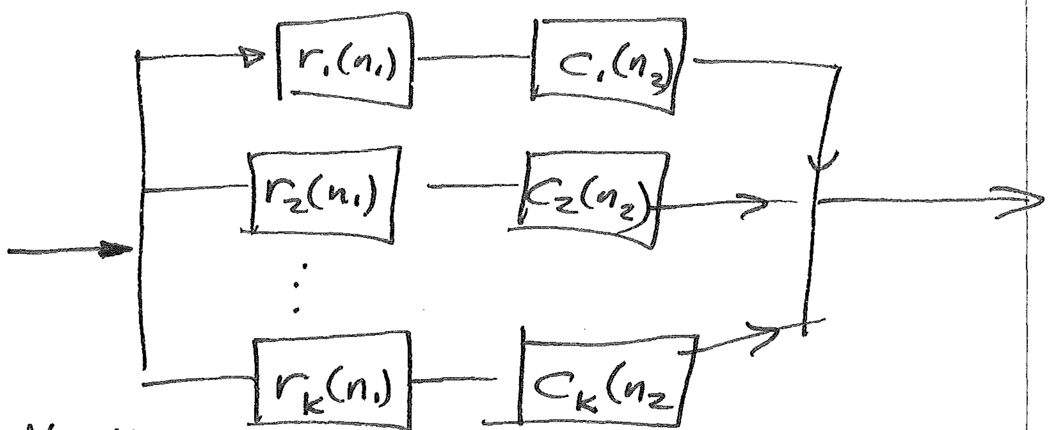
$$h[\vec{n}] = \frac{1}{|\det \underline{N}|} \sum_{\vec{k} \in R} I(2\pi \underline{N}^{-1} \vec{k})\, e^{j \vec{k}^T (2\pi \underline{N}^{-1}) \vec{n}}$$

3.5.2. PARALLEL FIR filters

Multistage Separable Filters

Recall: If $h[n_1, n_2]$ is FIR:

$$h[n_1, n_2] = \sum_{k=1}^{K} r_k(n_1)\, C_k(n_2)$$



$h(n_1, n_2) \in N_1 \times N_2$

Operations $= K(N_1 + N_2)$ multiplies
Conventional: $N_1 N_2$ "
} for a single output sample

Better if $K(N_1 + N_2) < N_1 N_2$

Ex: $N_1 = N_2 = K$

Regular better.

Optimal choice of decomposition

Research Open

Thus Far:
1. Restoring Lost Samples
2. Rotated Spectrum
3. Filter decomposition

3.5.3. Design of FIR filters using transformations

1-D to multi-dimensional xformation

1-D zero phase

$$H(\omega) = \sum_{n=-N}^{N} h[n] e^{-j\omega n}$$

$$= h[0] + \sum_{n=1}^{N} h[n] \left(e^{j\omega n} + e^{-j\omega n}\right)$$

$$= h[0] + 2 \sum_{n=1}^{N} h[n] \cos(\omega n)$$

$$= \sum_{n=0}^{N} a[n] \underbrace{\cos(\omega n)}_{\phi_n(\omega)}$$

$$a[n] = \begin{cases} h[0] & ; n = 0 \\ 2h[n] & ; 1 \le n \le N \end{cases}$$

IMPLEMENTATION OF FILTERS FROM TRANSFORMS

(Can use direct convolution, DFT, etc)

Use Transform Structure:

$$H(\vec{\omega}) = \sum_{n=0}^{N} a[n]\, T_n\left[F(\vec{\omega})\right]$$

Recurrance Relationship for Chebyshev:

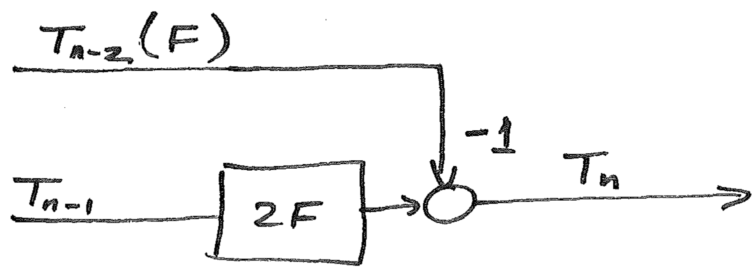
$$T_0(x) = 1$$
$$T_1(x) = x$$
$$T_n[x] = 2x\, T_{n-1}(x) - T_{n-2}(x)$$

Boils to Trig Identity

$$\cos n\omega = 2\cos\omega \cos(n-1)\omega - \cos(n-2)\omega$$

Anyway:

$$T_n\left(F(\vec{\omega})\right) = 2F(\vec{\omega})\, T_{n-1}\left(F(\vec{\omega})\right) - T_{n-2}\left(F(\vec{\omega})\right)$$



F is, for example, 3×3 fir filter

~~Impleme~~

Chebychev Polynomials

$$\cos wn = T_n[\cos w]$$

$n^{\underline{th}}$ order chebychev polynomials

$$T_0[x]=1 \Rightarrow T_0(\cos w) = 1 = \cos nw$$

$$T_1[x]=x \Rightarrow T_1(\cos w) = \cos w = \cos nw$$

$$T_2[x] = 2x^2-1 \Rightarrow T_1(\cos w) = 2\cos^2 w - 1$$

$$= 2\left[\tfrac{1}{2}\cos(2w) + \tfrac{1}{2}\right] - 1$$

$$= \cos 2w$$

etc.

Thus:

$$H(\omega) = \sum_{n=0}^{N} a[n]\, T_n(\cos \omega)$$

Multidimensional (McClellan transform)

$$H(\vec{\omega}) = \sum_{n=0}^{N} a[n]\, T_n(F(\vec{\omega}))$$

$F(\vec{\omega})$ should ~~be "close" to an M-D "equivalent" of "$\cos w$) & simple~~

should be a freq. response itself

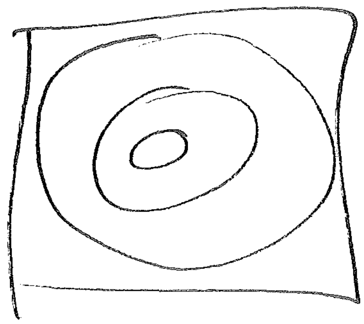
Ex in 2-D:

$$F(\omega_1,\omega_2) = A + B\cos\omega_1 + C\cos\omega_2 + D\cos(\omega_1-\omega_2) + E\cos(\omega_1+\omega_2)$$

ABCDE are free parameters.

~~Why e~~

Isopotentials:



Like contour Map.

~~Them~~: ~~th~~ Any Isopential of $F(\vec{\omega})$ is an isopotential of $H(\vec{\omega})$. ~~the~~

~~The~~

Proof:

$F(\vec{\omega})$ has isopotential

$F^n(\vec{\omega})$ " same "

$\sum_n b_n F^n(\vec{\omega})$ " " "

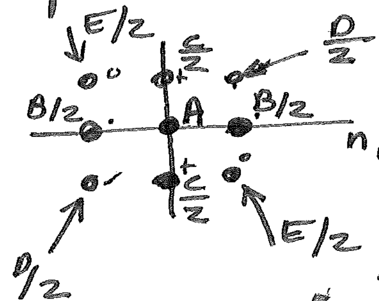
Since

$$H(\vec{\omega}) = \sum_n b_n F^n(\vec{\omega})$$

has same isopotential.

The <u>values</u> of the isopotentials are a function of ~~th~~ F <u>and</u> a[n]'s.

Ex Simple 3×3



← IF FIR

$$f[n_1,n_2] = A\,\delta(n_1,n_2) + \frac{B}{2}\left[\delta[n_1-1] + \delta[n_1+1]\right]\delta[n_2]$$
$$+ \frac{C}{2}\left[\qquad\qquad\right]$$

$\Updownarrow$

$$F(\omega_1,\omega_2) = A + B\cos\omega_1 + C\cos\omega_2$$
$$+ D\cos(\omega_1-\omega_2) + E\cos(\omega_1+\omega_2)$$

Ready to give out

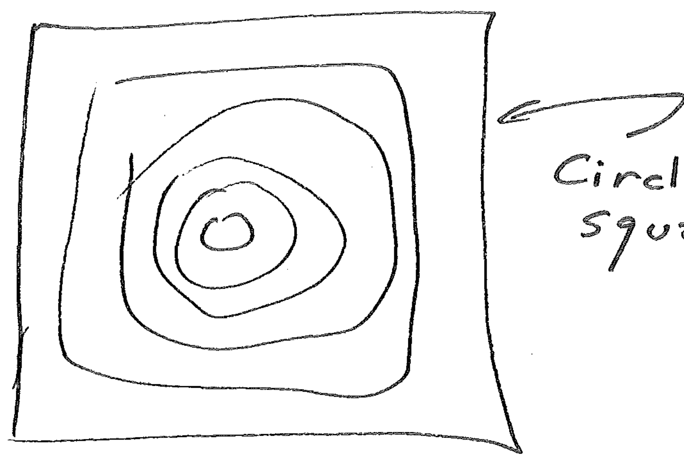
Design Procedure:

1. Fix isopotentials shapes by choosing $F(\vec{\omega})$
2. " " values by choosing $H(\vec{\omega})$.

Example:

Choose $A = -\frac{1}{2}$, $B = C = \frac{1}{2}$, $D = E = \frac{1}{4}$

Yields



Note: $F(\omega_1, \omega_2) = \frac{1}{2}[-1 + \cos\omega_1 + \cos\omega_2 + \cos\omega_1 \cos\omega_2]$

$F(\omega_1, 0) = \cos\omega_1$

Thus:

$$H(\omega_1, \omega_2) = \sum_{n=0}^{N} a[n]\, T_n[F(\omega_1, \omega_2)]$$

$$H[\omega_1, 0] = \sum_{n=0}^{N} a[n]\, T_n[F(\omega_1, 0)]$$

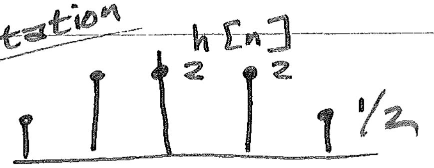
$$T_n[\cos\omega_1]$$

$$= H(\omega)$$

Nice "rotated" kind of spectrum. Choose $H(\omega)$ ∋ $H(\omega_1, \omega_2)$ is pseudo-rotated version (see p 141-142)

Use Low Pass or other

Ex Implementation



$$H(\omega) = \sum_{n=0}^{2} a[n] \cos n\omega$$

$$= 2 + 4\cos\omega + \cos 2\omega$$

$$= 2T_0(\omega) + 4T_1(\omega) + T_2(\omega)$$

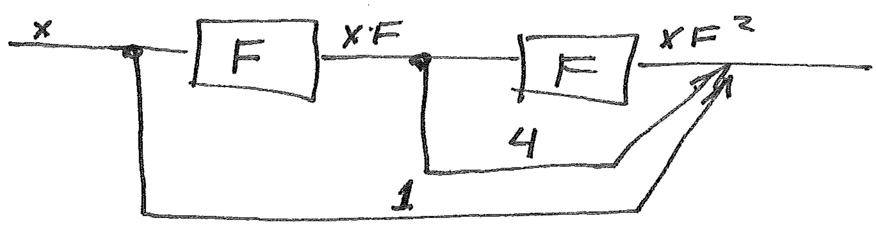
For a given $F(\omega_1, \omega_2)$

$$H(\omega_1, \omega_2) = 2T_0(F) + 4T_1(F) + T_2(F)$$

~~(Use i~~

$$= 2F^2(\omega_1, \omega_2) + 4F(\omega_1, \omega_2) + 1$$

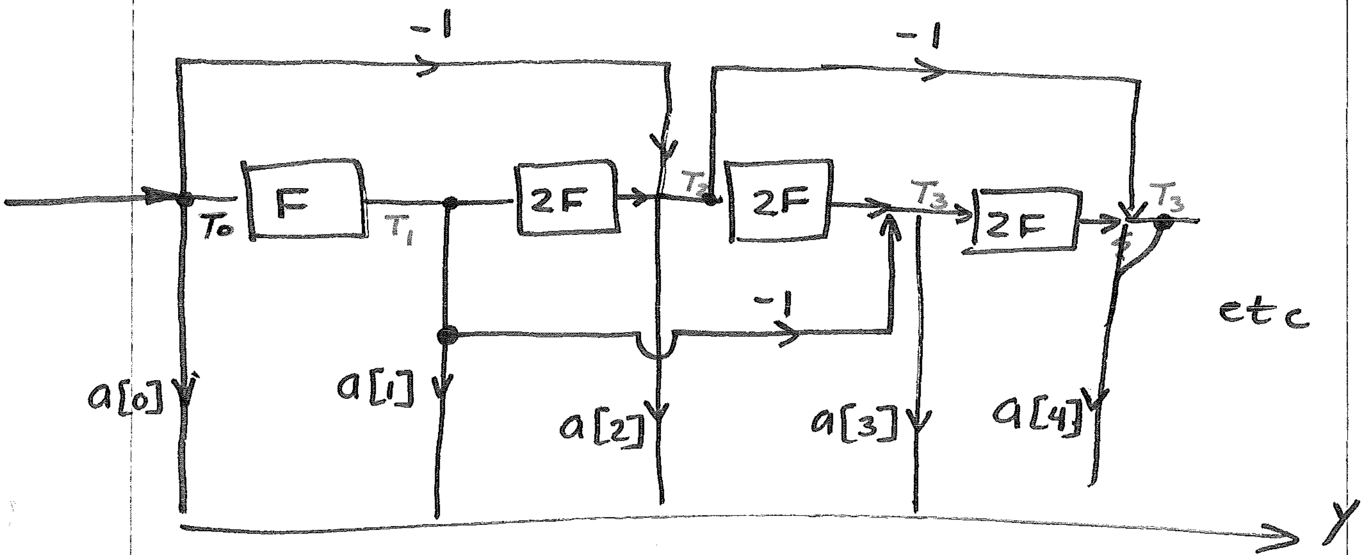
Realize



If $f[n_1, n_2]$ is FIR, so will be $h[n_1, n_2]$, though larger support. (elaborate).

For 3×3 zero phase F (2-D filter)

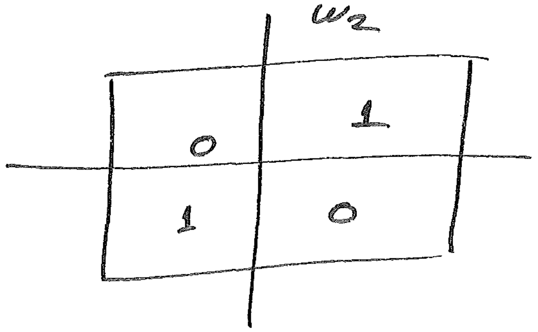
$$H(\vec{\omega}) = \sum_{n=0}^{N} a[n]\, T_n[F(\vec{\omega})]$$



F is simple (e.g. 3×3) FIR filter

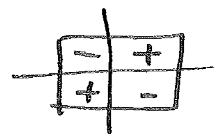
<u>Modular</u>

Example: FAN Filter



Step 1: Choose good contours

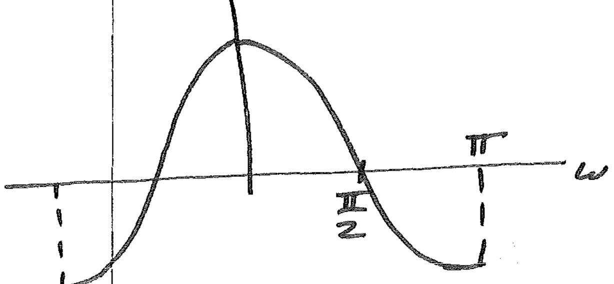
$F(\omega_1, \omega_2) = \sin\omega_1 \sin\omega_2$



$(A = B = C = 0,\ D = \frac{1}{2},\ E = -\frac{1}{2}]$

Substitution is

$\cos\omega = F(\omega_1, \omega_2) = \sin\omega_1 \sin\omega_2$



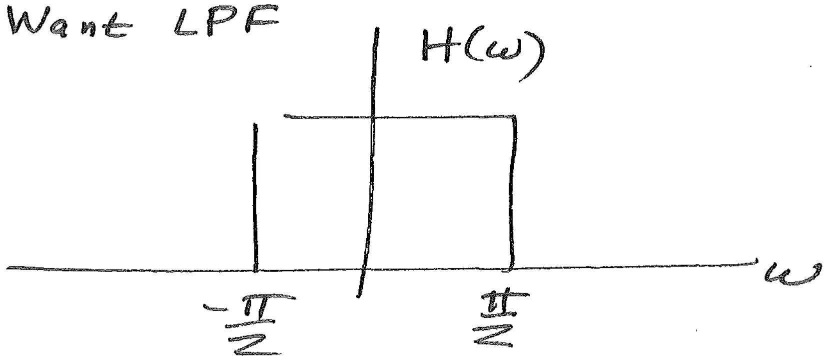
$H(\omega) = \sum a[n]\, T_n[\cos\omega]$

$H(\omega_1, \omega_2) = \sum a[n]\, T_n[\sin\omega_1 \sin\omega_2]$

Pos $(|\omega| < \frac{\pi}{2})$ mapped into quadrants I & III

Neg " " " II & III

Want LPF



Results shown on p.144
Fig 3.13

1.5.3. Alternate Definition of FT for Discrete Signals

Custom: $\underline{X}(\vec{\omega}) = \sum_{\vec{n}} x[\vec{n}]\, e^{-j\vec{\omega}^T\vec{n}}$

Recall $\vec{\omega} = \underline{V}^T \vec{\Omega}$

Gives Alternate:

$$\underline{X}_V(\vec{\Omega}) \triangleq \sum_{\vec{n}} x[\vec{n}]\, e^{-j\vec{\Omega}^T\underline{V}\vec{n}} = \underline{X}(\underline{V}^T\vec{\Omega})$$

Inverse Transform:

$$x[\vec{n}] = \frac{1}{(2\pi)^M}\int_{\mathcal{F}} \underline{X}(\vec{\omega})\, e^{j\vec{\omega}^T\vec{n}}\, d\vec{\omega}$$

$$\vec{\omega} = \underline{V}^T\vec{\Omega}$$

$$x[\vec{n}] = \frac{|\det \underline{V}|}{(2\pi)^M}\int_B \underline{X}(\underline{V}^T\vec{\Omega})\, e^{j\vec{\Omega}^T\underline{V}\vec{n}}\, d\vec{\Omega}$$

$$= \frac{(\det \underline{V})}{(2\pi)^M}\int_B \underline{X}_V(\vec{\Omega})\, e^{j\vec{\Omega}^T\underline{V}\vec{n}}\, d\vec{\Omega}$$

B = map of $\mathcal{F}$ in ω by $\vec{\omega} = \underline{V}^T\vec{\Omega}$ to $\vec{\Omega}$

As before $\underline{X}_V(\vec{\Omega})$ is periodic

$\underline{U}$ = periodicity matrix

$\underline{U}^T\underline{V} = 2\pi\underline{I}$

Proof:

$$\underline{X}_V(\vec{\Omega} + \underline{U}\vec{k}) = \sum_{\vec{n}} x[\vec{n}]\, e^{-j\vec{\Omega}^T\underline{V}\vec{n}}\, \underbrace{e^{-j\vec{k}^T\underline{U}^T\underline{V}\vec{n}}}_{=1}$$

Corresponding Theorems on p.51

3.6. Freq. Response of Hex FIR filters

~~For hex:~~ $\cancel{\underline{V} = c\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 1 & -1 \end{bmatrix}}$

Alternate $\mathcal{F}$ in sec. 1.5.3:

$$X_V(\vec{\Omega}) = \sum_{\vec{n}} x[\vec{n}]\, e^{-j\vec{\Omega}^T \underline{V}\vec{n}}$$

or, if $\underline{V}$ is dimensionless $\cancel{(c=1)}$

$$X_V(\vec{\omega}) = \sum_{\vec{n}} x[\vec{n}]\, e^{-j\vec{\omega}^T \underline{V}\vec{n}}$$

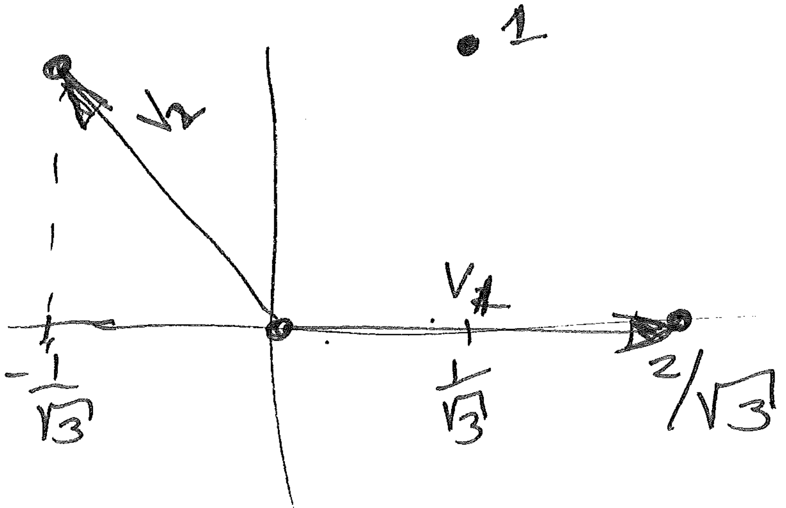
$$\vec{\omega}^T \underline{V}\vec{n} = [\omega_1 \; \omega_2]\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 1 & -1 \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$$

Consider Hex:

$$\underline{V} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 1 & -1 \end{bmatrix}$$

Note = det

$$\underline{OR} \quad \underline{V} = \begin{bmatrix} \frac{2}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ 0 & 1 \end{bmatrix}$$



Then

$$X_v(\vec{\omega}) = \sum_{\vec{n}} x[\vec{n}]\, e^{-j[\omega_1\, \omega_2]\begin{bmatrix} \frac{2}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \end{bmatrix}}$$

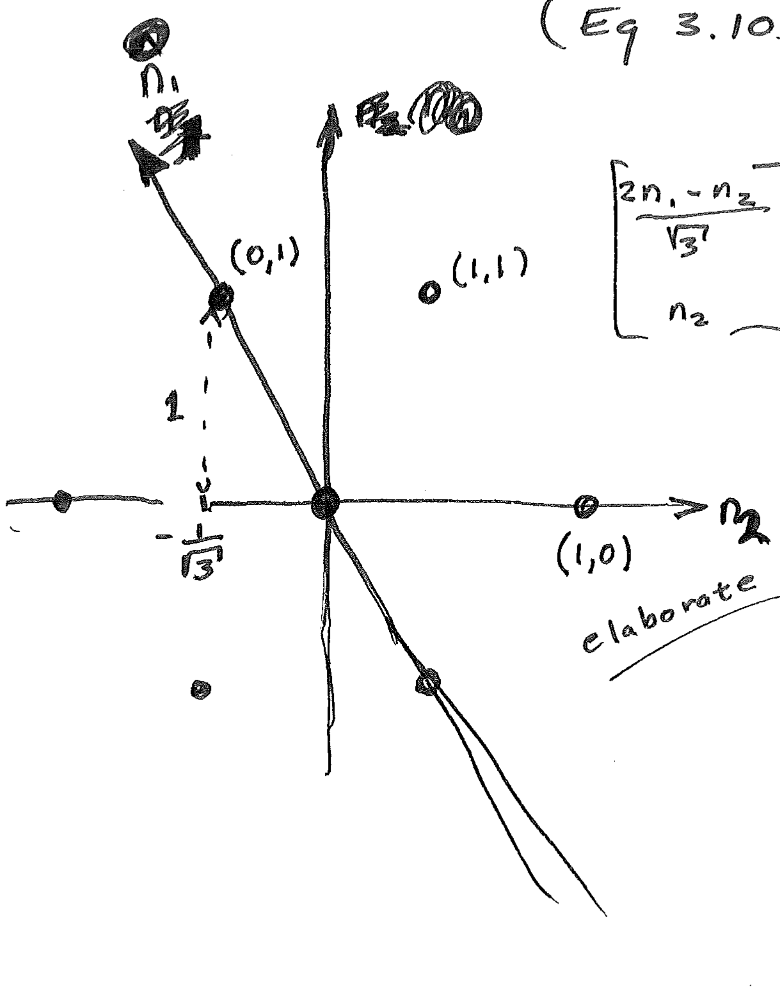
$$= \sum_{\vec{n}} x[\vec{n}]\, e^{-j[\omega_1\ \omega_2]\begin{bmatrix} \frac{2n_1 - n_2}{\sqrt{3}} \\ n_2 \end{bmatrix}}$$

$$= \sum_{\vec{n}} x[\vec{n}]\, e^{-j\left[\left(\frac{2n_1 - n_2}{\sqrt{3}}\right)\omega_1 + n_2\omega_2\right]}$$

Thus, for hex filter, $x = h$ ⇒

$$H(\omega_1, \omega_2) = \sum_{n_1, n_2} x[n_1, n_2]\, e^{-j\left[\left(\frac{2n_1 - n_2}{\sqrt{3}}\right)\omega_1 + n_2\omega_2\right]}$$

(Eq 3.103).



$$\begin{bmatrix} \frac{2n_1 - n_2}{\sqrt{3}} \\ n_2 \end{bmatrix}$$

elaborate

Let x & h be hex

If $Y_v(\vec{\omega}) = X_v(\vec{\omega}) H_v(\vec{\omega})$

Then

$$y[\vec{n}] = \sum_{\vec{k}} h[\vec{k}]\, x[\vec{n} - \vec{k}]$$

Proof Like always.

3.6.2. Design of Hex Filters

Windows:

$$h[\vec{n}] = i[\vec{n}]\, w[\vec{n}]$$

Choose w as before

For 1-D window, outer product becomes:

$$w[n_1, n_2] = v[n_1]\, v[n_2]\, v[n_1 - n_2]$$

Note: If $v[n]$ has finite support, then w has hex support

For continuous case:

$$w[t_1\ t_2] = v(t_1)\, v(t_2)\, v(t_1 - t_2)$$

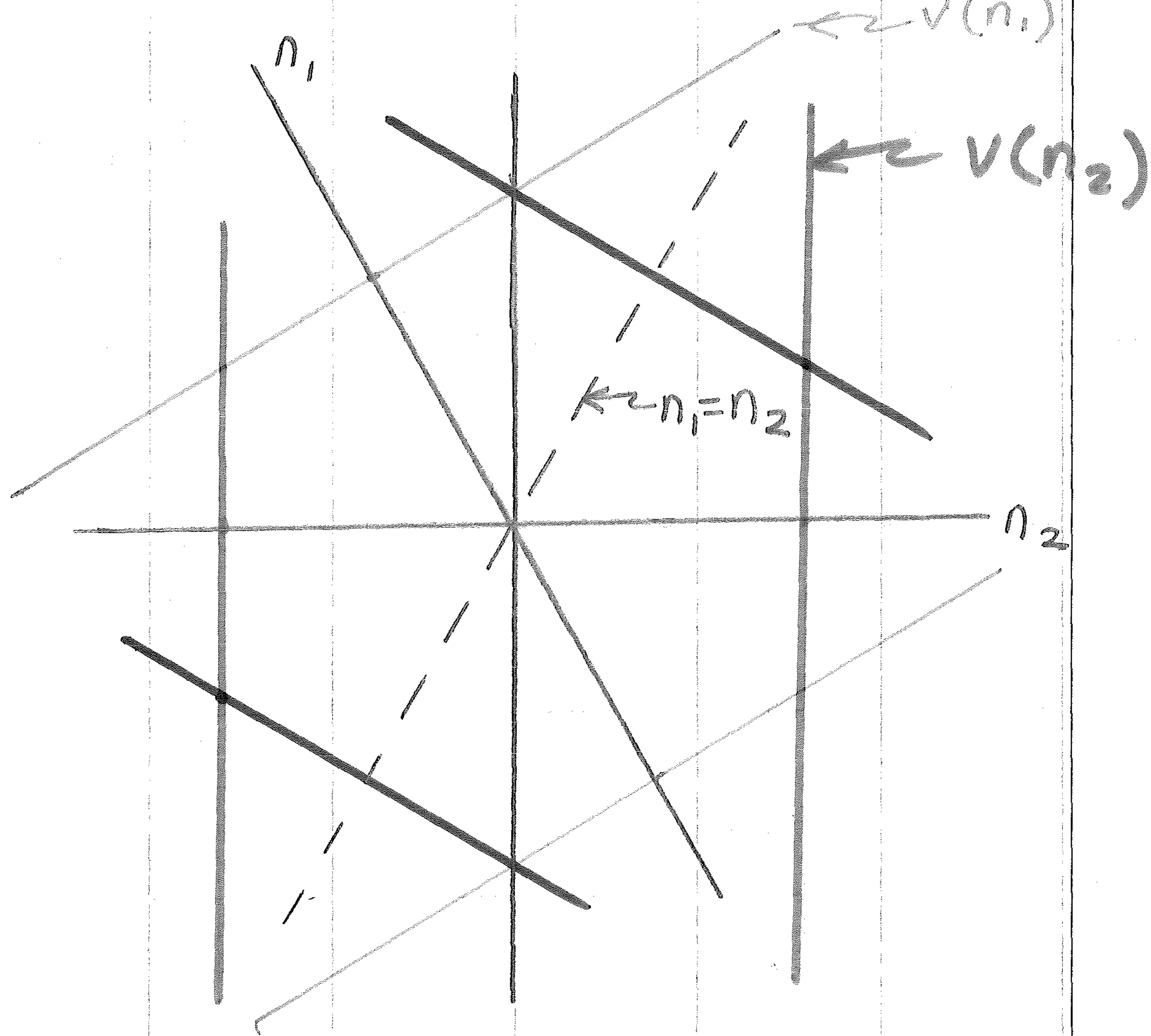
$v(t) = 0$ for $|t| > \tau$ $\qquad -\tau < t < \tau$

$v(t_1 - t_2) = 0$ for $|t_1 - t_2| > \tau$

$$-\tau < t_1 - t_2 < \tau$$

$$-\tau - t_1 < -t_2 < \tau - t_1$$

$$t_1 - \tau < t_2 < t_1 + \tau$$

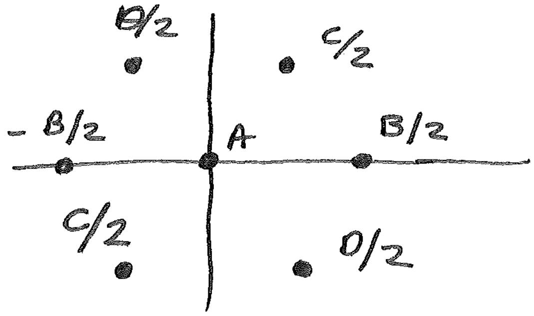


n_1
$v(n_1)$
$v(n_2)$
$n_1 = n_2$
n_2

"Rotated" window for hex

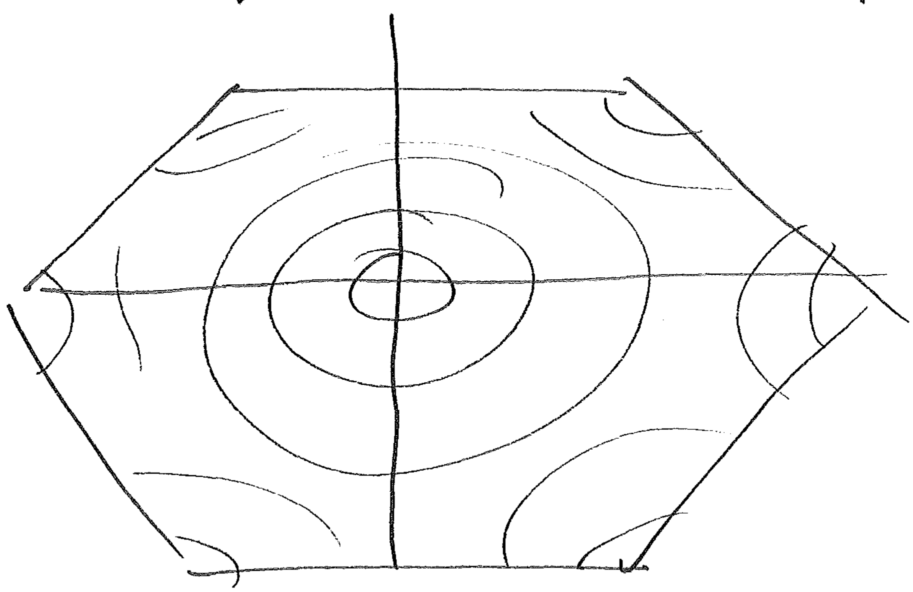
$$w(n_1, n_2) = v\Big(\underbrace{\tfrac{2}{\sqrt{3}}\sqrt{n_1^2+n_2^2-n_1 n_2}}_{\text{Circle in distorted coordinate system.}}\Big)$$

McClellan Transform for HEX filters



$$F(\omega_1,\omega_2) = A + B\cos\frac{2\omega_1}{\sqrt{3}} + C\cos\left(\frac{\omega_1}{\sqrt{3}} + \omega_2\right) + D\cos\left(\frac{\omega_1}{\sqrt{3}} - \omega_2\right)$$

For $A = -\frac{1}{3}$, $B = C = D = 4/9$, Isopotentials are



p. 153 Bottom

Good for rotated spectrum when on circle.

4. MULTIDIMENSIONAL DIFFERENCE EQUATIONS

$y[\vec{n}]$ = output

$x[\vec{n}]$ = input

$$* \quad \sum_{\vec{k}} b[\vec{k}]\, y[\vec{n}-\vec{k}] = \sum_{\vec{r}} a[\vec{r}]\, x[\vec{n}-\vec{r}]$$

FINITE ORDER SYSTEM: SUMS ARE FINITE

FILTER ORDER: SIZE OF SUPPORT FOR $b[\vec{k}]$

For example, for

$$\vec{k} \in \underbrace{N_1 \times N_2 \times \ldots \times N_M}_{\text{FILTER ORDER}}$$

IMPORTANT SPECIAL CASE: zero ORDER

$$y[\vec{n}] = \sum_{\vec{r}} a[\vec{r}]\, x[\vec{n}-\vec{r}]$$

= CONVOLUTION

= FIR Filter

Note * can be written as:

$$b(\vec{0})\, y[\vec{n}] + \sum_{\vec{k}\neq\vec{0}} b[\vec{k}]\, y[\vec{n}-\vec{k}]$$

$$= \sum_{\vec{r}} a[\vec{r}]\, x[\vec{n}-\vec{r}]$$

NORMALIZE $b(\vec{0}) = 1$. Then:

$$y[\vec{n}] = \sum_{\vec{r}} a[\vec{r}]\, x[\vec{n}-\vec{r}] - \sum_{\vec{k}\neq 0} b[\vec{k}]\, y[\vec{n}-\vec{k}]$$

COMPUTE FROM INPUT & "PREVIOUS" SAMPLES

ASSUMPTION: Required $x[\vec{n}]$ is available

" " $y[\vec{n}]$ has already been computed or are available from initial conditions

IF SO, SYSTEM IS "RECURSIVELY COMPUTABLE"

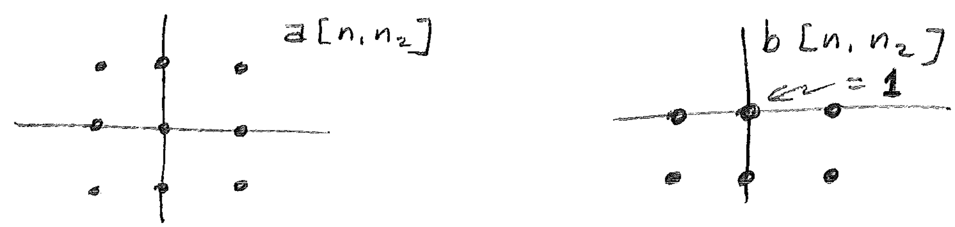
4.1.2. Recursive Computability

In 2-D, we can write:

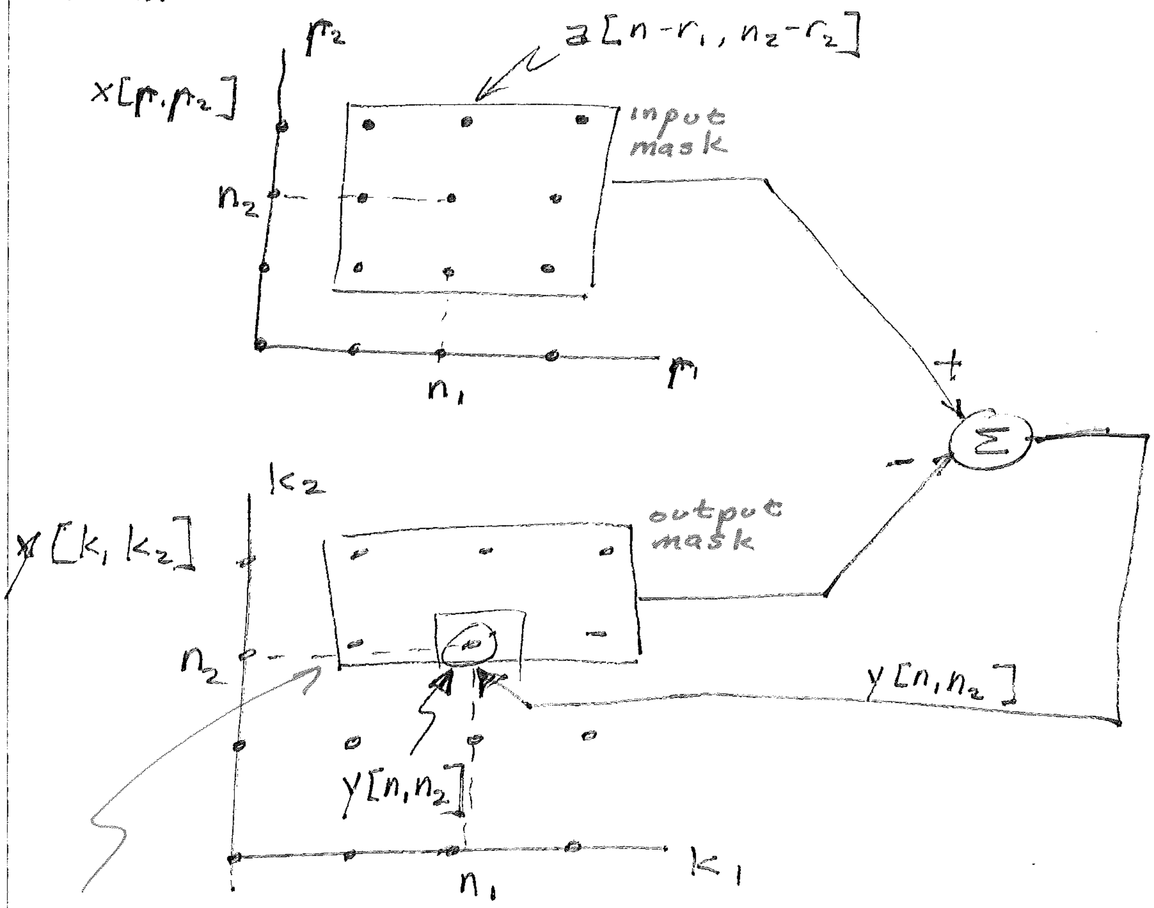
$$y[n_1 n_2] = \sum_{r_1} \sum_{r_2} a(n_1 - r_1, n_2 - r_2)\, x(r_1 r_2)$$
$$- \sum_{k_1} \sum_{k_2} b(n_1 - k_1, n_2 - k_2)\, y(k_1 k_2)$$

= SUM OF CONVOLUTIONS

Ex



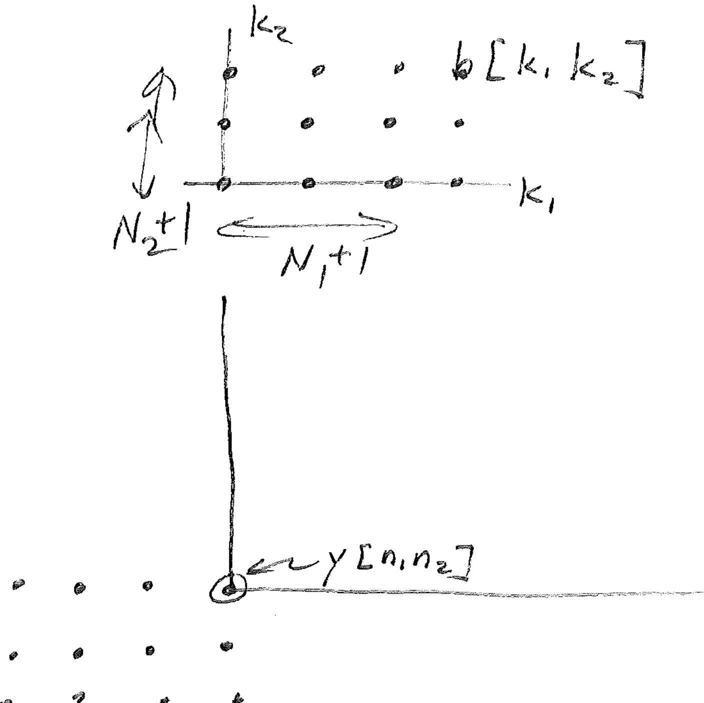
Then



Must cover known samples ⇒ determines recursive computibility

(First quadrant)

"Causal" $b[k_1, k_2]$ is recursively computable

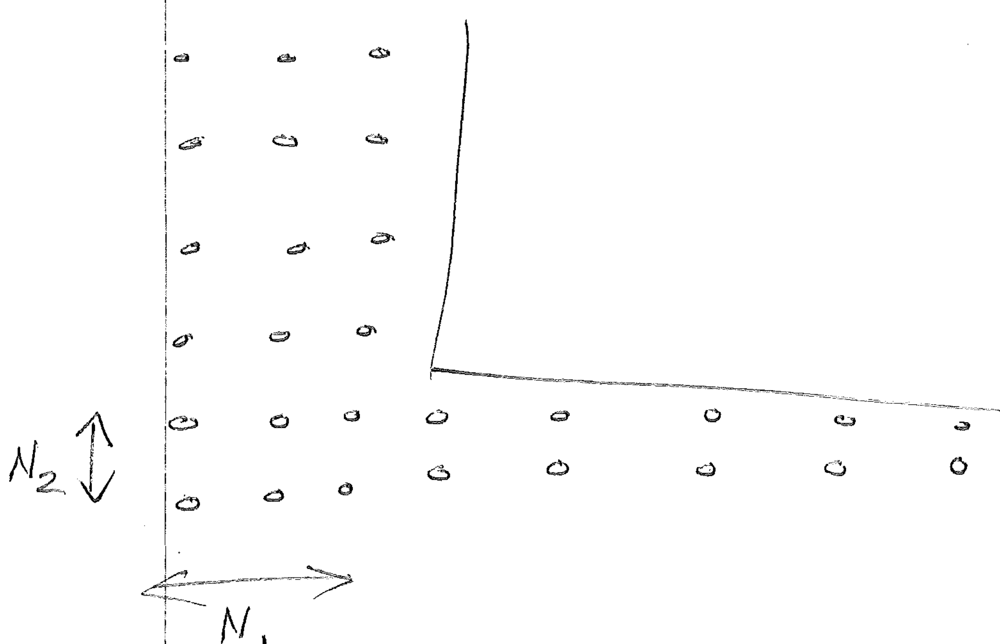


~~"Required!~~

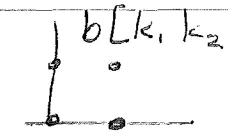
Must have appropriate boundry:

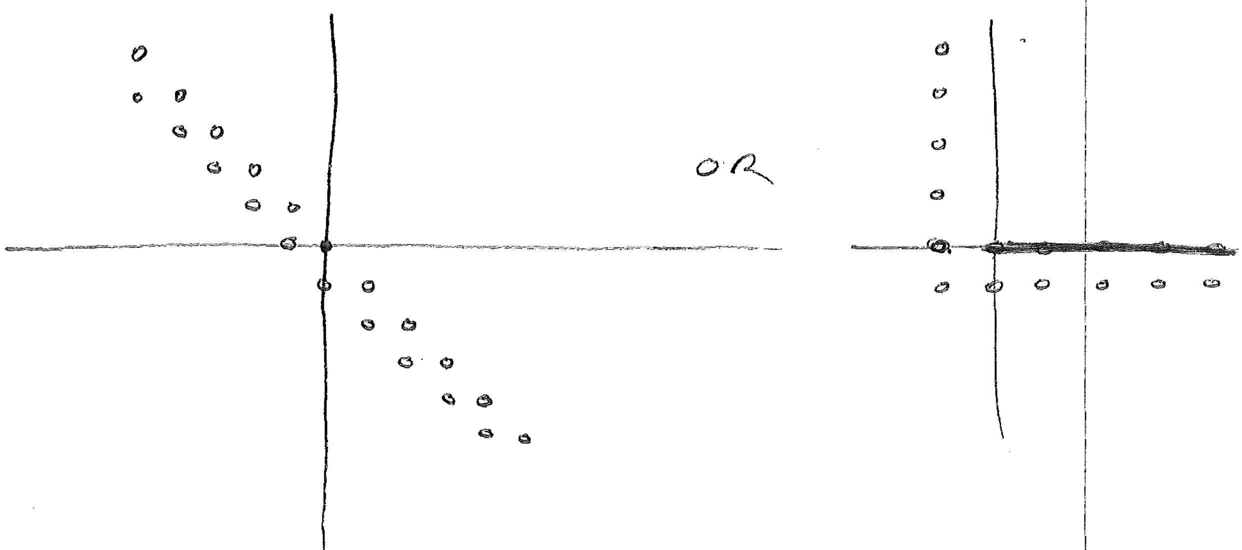
(Can fill entire plane)

(elaborate



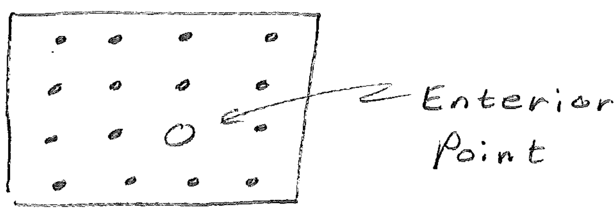
Alternate:

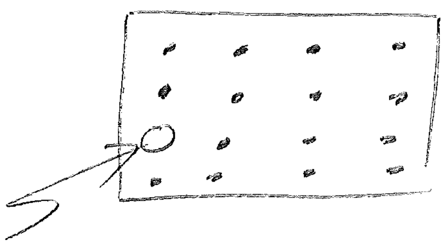




Examples of non recursively computable masks

b's





Non corner

Boundry Conditions

For difference equation:

$$y = y_h + y_p$$

y_h = RESPONSE TO INITIAL (BOUNDRY) CONDITIONS

y_p = FORCED RESPONSE (DUE TO x)

For $y_h = 0$, MUST HAVE ZERO BOUNDRY CONDITIONS

But, it makes a difference where you put the B.C. For example

$$y[n_1, n_2] = y[n_1 - 1, n_2] + y[n_1 + 1, n_2 - 1] + x[n_1, n_2]$$

Set 1 of B.C. on p.170

Response to $\delta[n_1, n_2]$ & $\delta[n_1 - 1, n_2 - 1]$

Note: **Not** shift invariant

Set 2 of B.C. on p.171

Here, looks shift invariant.

Q: How Do We Choose Location of B.C.?

A: Outside of support of y

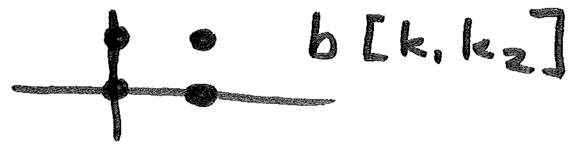
For filters of finite order, use a "V". Exact orientation is a function of output mask.

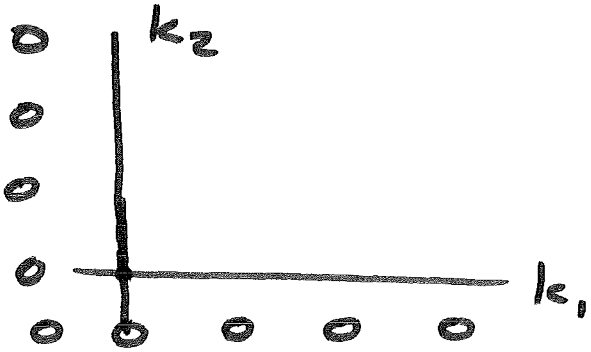
~~Otherwise, vague (Use z xform?)~~

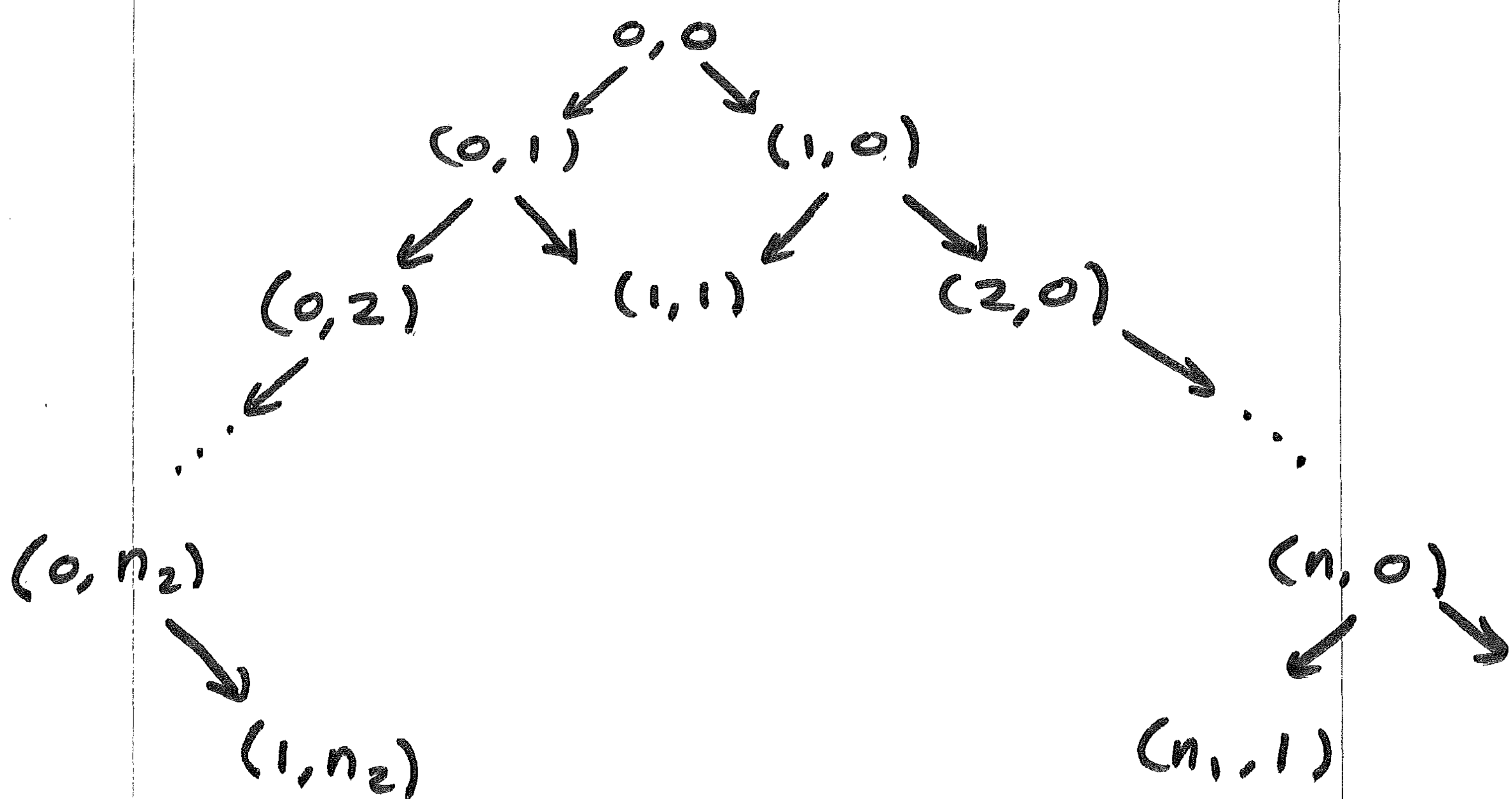
4.1.4. ORDERING THE COMPUTATION OF OUTPUT SAMPLES

In 1-D, There is no choice in computed order

Ex: FIRST Quadrant Example







PRECIDENCE GRAPH

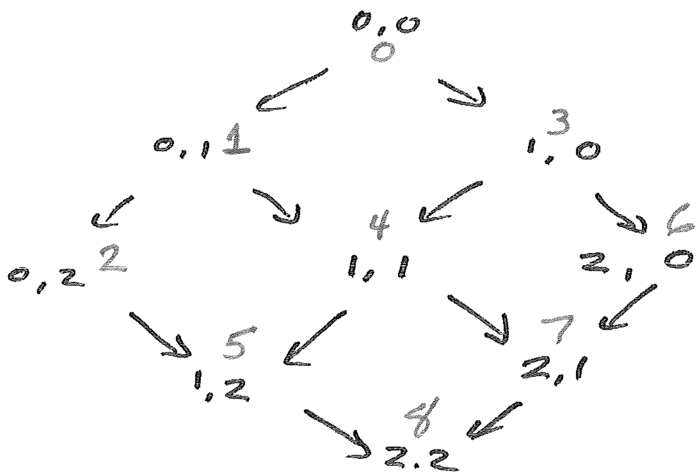
ORDERING:

ASSIGN EACH A #

$$n = I(\vec{n})$$

Ex: for $[0, N_1 - 1] \times [0, N_2 - 1]$

$$n = N_2 n_1 + n_2$$

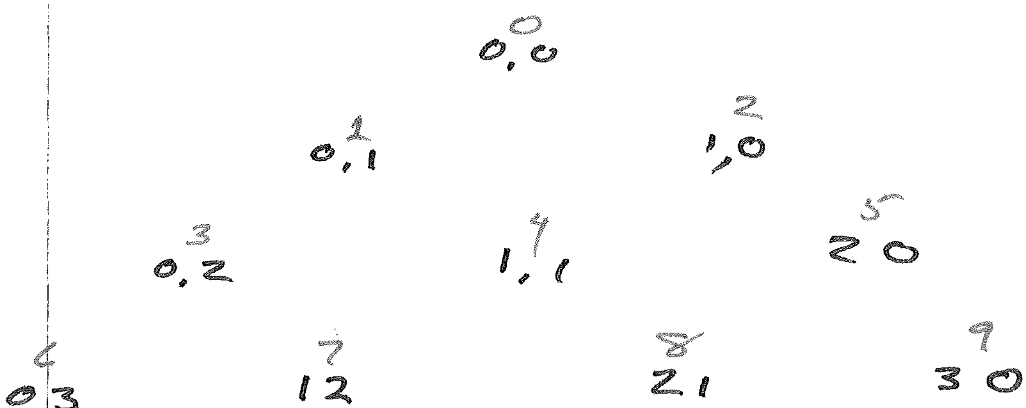


$N_1 = N_2 = 3$

$n = 3n_1 + n_2$

For row, let:

$$n = \tfrac{1}{2}(n_1 + n_2)(n_1 + n_2 + 1) + n_1$$

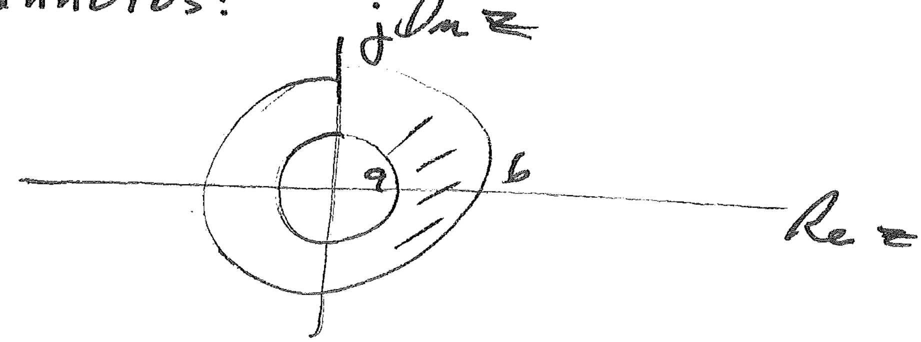


4.2. MULTIDIMENSIONAL z TRANSFORMS

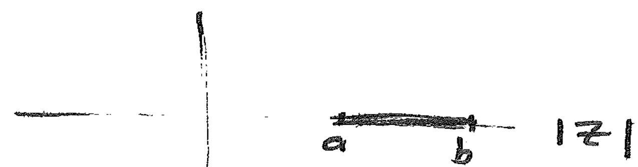
$$H_z(z_1, z_2) = \sum_{n_1, n_2} h[n_1, n_2]\, z_1^{-n_1} z_1^{-n_2}$$

$$H_z(e^{j\omega_1}, e^{j\omega_2}) = H(\omega_1, \omega_2) \Leftarrow$$ ASSUMING CONVERGENCE ON UNIT CIRCLE

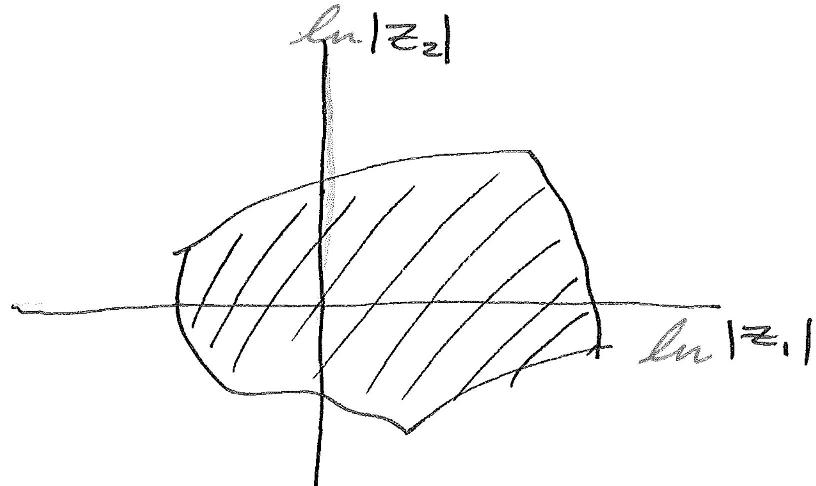
For 1-D case, Convergece in within an annulus:



Can translate this into 1-D interval

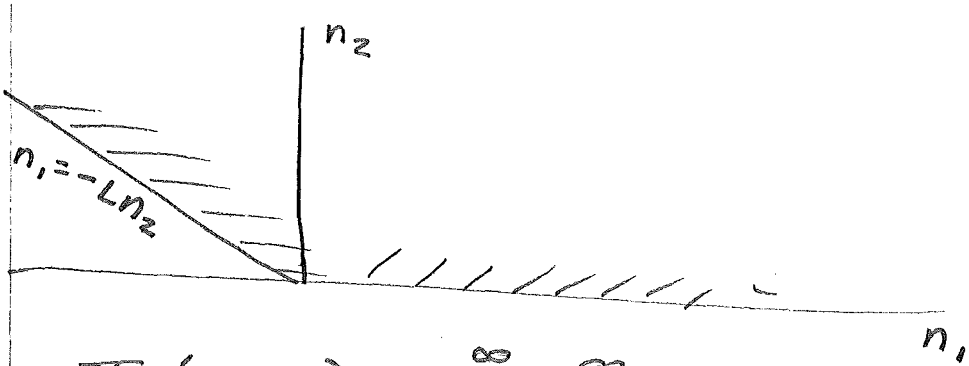


z_1 z_2 "plane" is a 4-D surface, but we can look at it in 2-D only



specifying $H_z(z_1, z_2)$ is ambiguous w/o specifying region of convergence.

3. Sequences with support on a wedge



$$X_z(z_1, z_2) = \sum_{n_2=0}^{\infty} \sum_{n_1=-Ln_2}^{\infty} x[n_1, n_2]\, z_1^{-n_1} z_2^{-n_2}$$

Let $\ell = n_1 + L n_2$

$$X_z(z_1 z_2) = \sum_{n_2=0}^{\infty} \sum_{\ell=0}^{\infty} x[\ell - Ln_2, n_2]\, z_1^{-\ell + Ln_2} z_2^{-n_2}$$

$$= \underbrace{\sum_{n_2=0}^{\infty} \sum_{\ell=0}^{\infty}}_{A} \underbrace{x[\ell - Ln_2, n_2]}_{\text{support in first quadrant}} z_1^{-\ell} \left(z_1^{-L} z_2\right)^{-n_2}$$

note

As before, if (z_{01}, z_{02}) has convergence, then (z_1, z_2) will also if

$$|z_1| \geq |z_{01}|$$

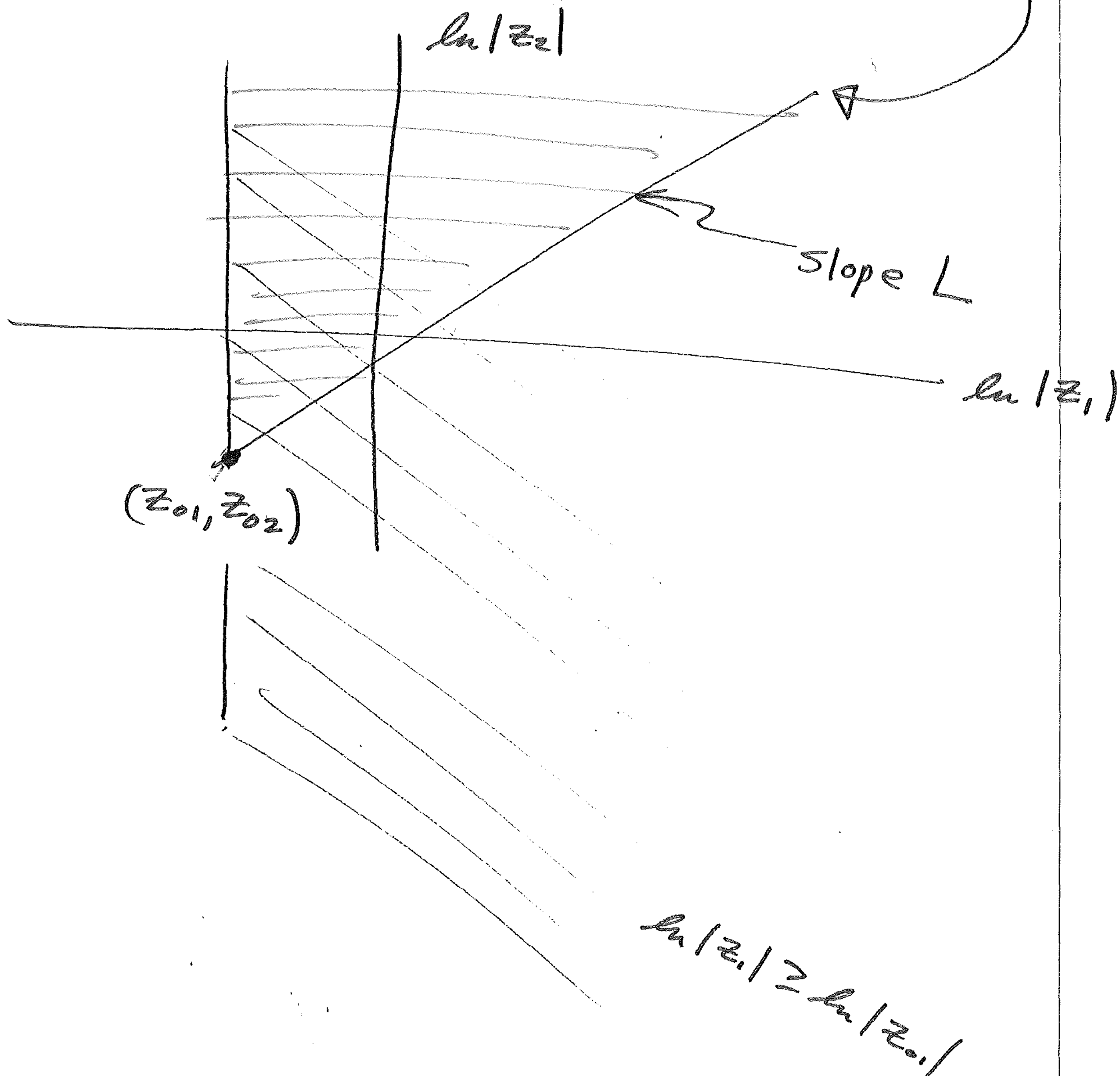
and $$|z_1^{-L} z_2| \geq |z_{01}^{-L} z_{02}|$$

or, taking $\ln$'s

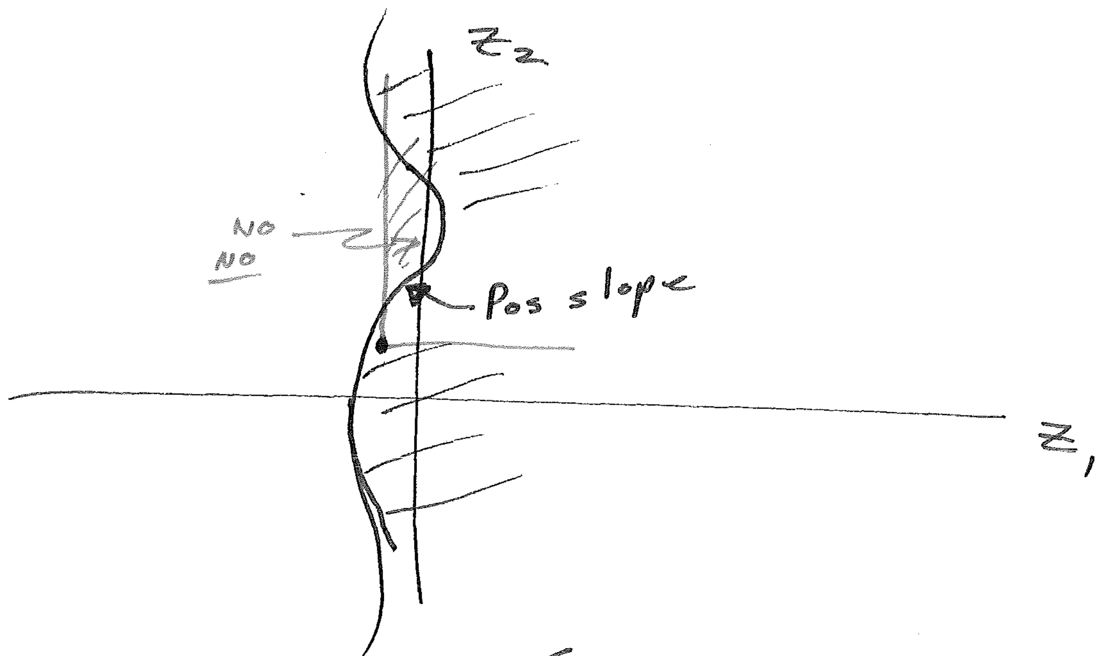
$$\ln|z_1| \geq \ln|z_{01}|$$

$$\ln|z_1|^{-L} + \ln|z_2| \geq \ln|z_{01}|^{-L} + \ln|z_{02}|$$

or $$\ln|z_2| \geq L\ln|z_1| + \{\ln|z_{01}| - L\ln|z_{01}|\}$$



A Conclusion: Boundry of region of convergence must have nonpositive (ie, negative or zero) slope.



Ex $x[n_1 n_2] = a^{n_1} \delta[n_1 - n_2] u[n_1, n_2]$

$$X(z_1, z_2) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \delta[n_1 - n_2] \left(\frac{z_1}{a}\right)^{-n_1} z_2^{-n_2}$$

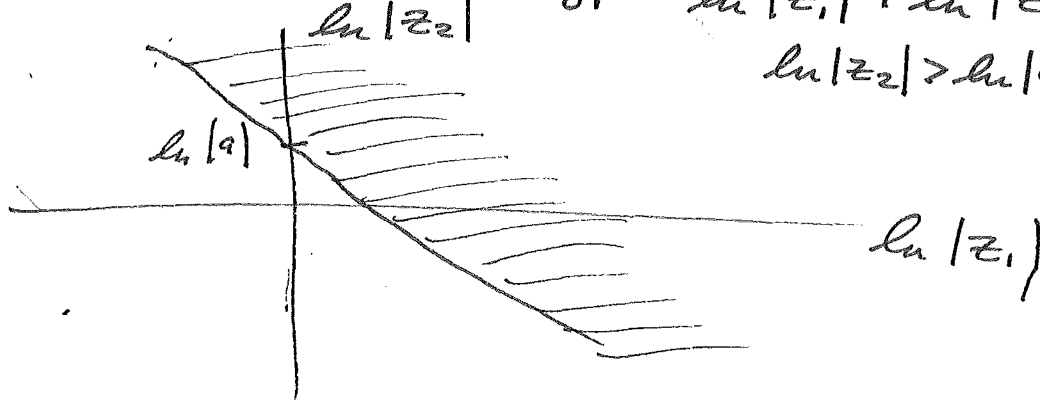
$$= \sum_{n_1=0}^{\infty} \left(\frac{z_1 z_2}{a}\right)^{-n_1} = \sum_{n=0}^{\infty} \left(\frac{a}{z_1 z_2}\right)^{n}$$

$$= \frac{1}{1 - a z_1^{-1} z_2^{-1}} \quad ; \quad \left|\frac{a}{z_1 z_2}\right| < 1$$

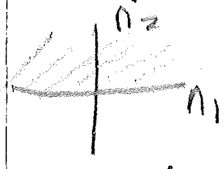
$|z_1 z_2| > |a|$

or $\ln|z_1| + \ln|z_2| > \ln|a|$

$\ln|z_2| > \ln|a| - \ln|z_1|$



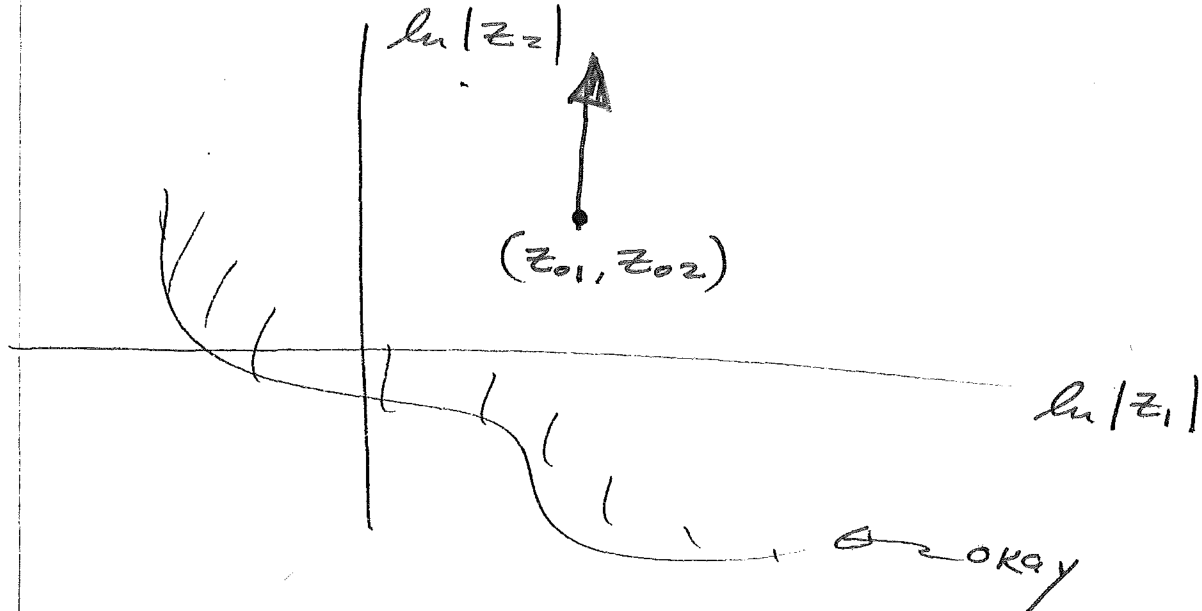
4. Sequences with support on a half plane



$$x[n_1, n_2] = x[n_1, n_2]\, u[n_2]$$

$$X_z(z_1, z_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=0}^{\infty} x[n_1, n_2]\, z_1^{-n_1} z_2^{-n_2}$$

Converges for (z_{01}, z_{02}), then
converges for $|z_1| = |z_{01}|$
and $|z_2| \geq |z_{02}|$



5. Sequences with support everywhere

Sometimes everywhere $e^{-n_1^2 - n_2^2}$
" nowhere $2^{|n_1|}\, 2^{|n_2|}$

Divide into 4 Quadrants.
Intersect regions of convergence

4.2.3. Properties of 2-D transforms

1. Separable Signals:

$$x[n_1 n_2] = v[n_1] w[n_2]$$

$$X_z(z_1 z_2) = V_z(z_1)\, W_z(z_2)$$

2. Linearity

3. Shift

$$x[n_1 n_2] = v[n_1 + m_1, n_2 + m_2]$$

$$X_z(z_1 z_2) = V(z_1 z_2)\, z_1^{m_1} z_2^{m_2}$$

4. Modulation

$$x[n_1 n_2] = a^{n_1} b^{n_2} w[n_1 n_2]$$

$$X_z(z_1 z_2) = W_z(a^{-1} z_1, b^{-1} z_2)$$

5. Differentiation

~~x[n, n~~

$$n_1 n_2\, w[n_1 n_2] \Longleftrightarrow z_1 z_2 \frac{\delta^2}{\delta z_1 \delta z_2} W_z(z_1 z_2)$$

6. Conjugation

$$x^*(n_1 n_2) \Longleftrightarrow X^*(z_1^*, z_2^*)$$

7. Reflection

$$x[-n_1\ n_2] \Longleftrightarrow X_z(z_1^{-1}, z_2)$$

8. Convolution

$$Y_z = X_z H_z$$

9. Initial Value them

x is first quadrant

$$\lim_{z_1 \to \infty} X_z(z_1, z_2) = \sum_{n_2} x[0, n_2] z_2^{-n_2}$$

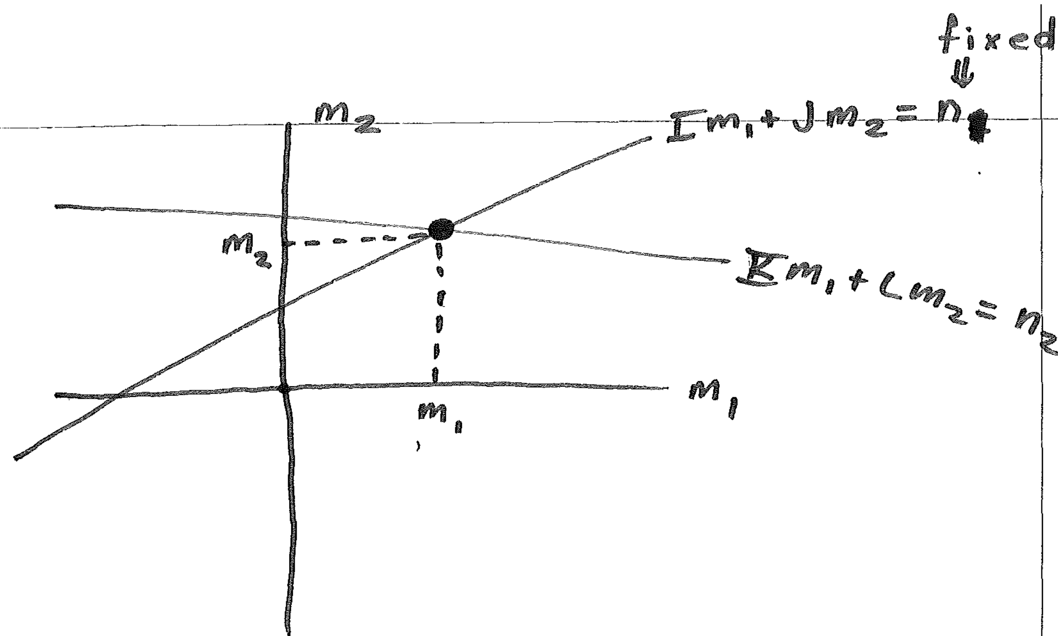
$$\lim_{\substack{z_1 \to \infty \\ z_2 \to \infty}} X_z(z_1 z_2) = x(0,0)$$

10. Linear mappings

$$x[n_1 n_2] = \begin{cases} w[m_1 m_2] \, ; & n_1 = Im_1 + Jm_2 \\ & n_2 = Km_1 + Lm_2 \\ 0 & ; ow \end{cases}$$

$IL - KJ \neq 0$

$$\Rightarrow X_z(z_1 z_2) = W_z(z_1^I z_2^K, z_1^J z_2^K$$



Proof:

$$X_z(z_1, z_2) = \sum_{n_1} \sum_{n_2} x[n_1, n_2] z_1^{-n_1} z_2^{-n_2}$$

$$= \sum_{m_1} \sum_{m_2} \underbrace{x[n_1, n_2]}_{w[m_1, m_2]} z_1^{-Im_1 - Jm_2} z_2^{-Km_1 - Lm_2}$$

$$= \sum_{m_1} \sum_{m_2} w[m_1, m_2] \left(z_1^I z_2^K\right)^{-m_1} \left(z_1^J z_2^L\right)^{-m_2}$$

$$= W_z\left(z_1^I z_2^K,\ z_1^J z_2^L\right)$$

~~4.2.3. Properties of 2-D Z transform (not in lecture)~~

4.2.4. Transfer function of systems specified by difference eqs.

2-D difference eq:

$$\sum_{k_1}\sum_{k_2} b[k_1,k_2]\, y[n_1-k_1, n_2-k_2]$$

$$= \sum_{r_1}\sum_{r_2} a[r_1,r_2]\, x[n_1-r_1, n_2-r_2]$$

A 2-D Z transform:

$$H_z(z_1,z_2) = \frac{Y(z_1,z_2)}{X(z_1,z_2)}$$

$$= \frac{\sum_{k_1}\sum_{k_2} a[k_1,k_2] z_1^{-k_1} z_2^{-k_2}}{\sum_{k_1}\sum_{k_2} b[k_1,k_2) z_1^{-k_1} z_2^{-k_2}}$$

$$= \frac{A_z(z_1,z_2)}{B_z(z_1,z_2)}$$

Poles if $A_z(z_1,z_2) \neq 0$ and $B \neq 0$

multidimensional poles are continuous

4.2.5. Inverse Z transform

$$x[n_1, n_2] = \frac{1}{(j2\pi)^2} \oint_{C_2} \oint_{C_1} X_z(z_1, z_2)\, z_1^{n_1-1} z_2^{n_2-1}\, dz_1 dz_2$$

ccw in z_2 plane

Easiest to use partial fractions

Aternate (overlooked) form

if converge there:

$$X_z(e^{j\omega_1} e^{j\omega_2}) = X(\omega_1, \omega_2)$$

$$x[n_1, n_2] = \frac{1}{(2\pi)^2} \iint_{-\pi}^{\pi} X_z(e^{j\omega_1} e^{j\omega_2})\, e^{j(\omega_1 n_1 + \omega_2 n_2)}\, d\omega_1 d\omega_2$$

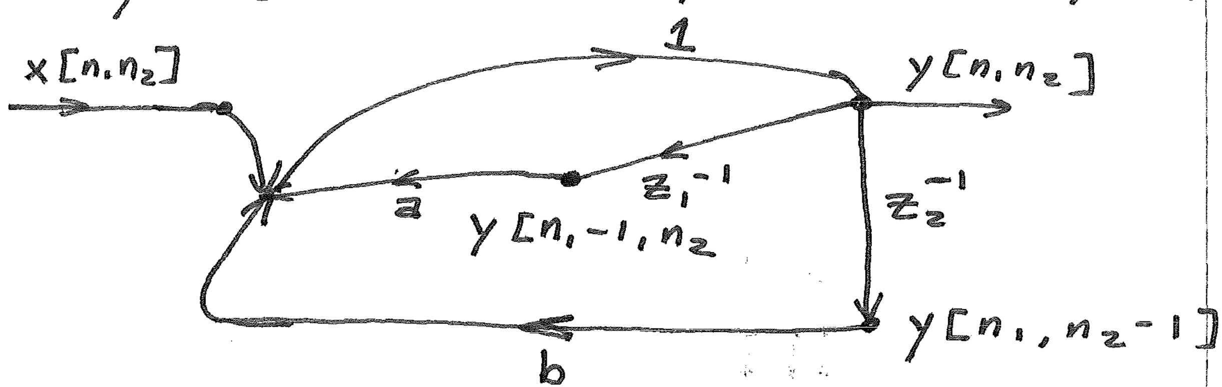
4.2.6. 2-D Flowgraphs

$\xrightarrow{c}$	gain	$c f[n_1,n_2]$
$\xrightarrow{z_1^{-1}}$	shift right	$f[n_1-1, n_2]$
$\xrightarrow{z_1}$	shift left	$f[n_1+1, n_2]$
$\xrightarrow{z_2^{-1}}$	shift up	$f[n_1, n_2-1]$
$\xrightarrow{z_2}$	shift down	$f[n_1, n_2+1]$

Ex $\quad H(z_1,z_2)=\dfrac{1}{1-az_1^{-1}-bz_2^{-1}} = \dfrac{Y}{X}$

from
$y[n_1,n_2]= x[n_1,n_2] + a\,y[n_1-1,n_2] + b\,y[n_1,n_2-2]$

FIG 4.21 is wrong



Can use Mason's Gain rule

Problem: Can't implement from flowgraph like in 1-D case. Must specify use ~~of~~ precidence graph

unless done in //

5. 2-D IIR Filter Design & Implementation

5.2. Iterative Implementations for 2-D IIR filters

$$H(\vec{\omega}) = \frac{A(\vec{\omega})}{B(\vec{\omega})}$$

$$A(\vec{\omega}) = \sum_{\vec{\ell}} a(\vec{\ell})e^{-j\vec{\omega}^T\vec{\ell}}; \quad B(\vec{\omega}) = \sum_{k} b(\vec{k})e^{j\vec{\omega}^T\vec{k}}$$

Define $\quad C(\vec{\omega}) \triangleq 1 - B(\vec{\omega})$

$$\Rightarrow H(\vec{\omega}) = \frac{A(\vec{\omega})}{1 - C(\vec{\omega})}$$

$$Y(\vec{\omega}) = H(\vec{\omega})X(\vec{\omega})$$
$$= \frac{A(\vec{\omega})X(\vec{\omega})}{1 - C(\vec{\omega})}$$

$\cancel{\text{or} \quad Y(\vec{\omega}) = A(\vec{\omega})X(\vec{\omega}) + C(\vec{\omega})Y(\vec{\omega})}$

$\cancel{\text{or} \quad y[\vec{n}] = a[\vec{n}] * x[\vec{n}] + c[\vec{n}] * y[\vec{n}]}$

If $\quad |C(\vec{\omega})| < 1$

$$\frac{1}{1 - C(\vec{\omega})} = \sum_{k=0}^{\infty} C^k(\vec{\omega})$$

And

$$Y(\vec{\omega}) = \sum_{k=0}^{\infty} C^k(\vec{\omega})\, A(\vec{\omega})X(\vec{\omega})$$

Define

$$Y_N(\vec{\omega}) = \sum_{k=0}^{N} C^k(\omega)A(\vec{\omega})X(\vec{\omega})$$

Better form:

$$Y_N = \sum_{k=0}^{N} C^k A \underline{X}$$

$$= Ax + \sum_{k=1}^{N} C^k A \underline{X}$$

set $\ell = k-1$

$$Y_N = A\underline{X} + \sum_{\ell=0}^{N-1} C^{\ell+1} A \underline{X}$$

$$= A\underline{X} + C \sum_{\ell=0}^{N-1} C^{\ell} A \underline{X}$$

$$\underline{Y}_N = A\underline{X} + C\,\underline{Y}_{N-1}$$

or

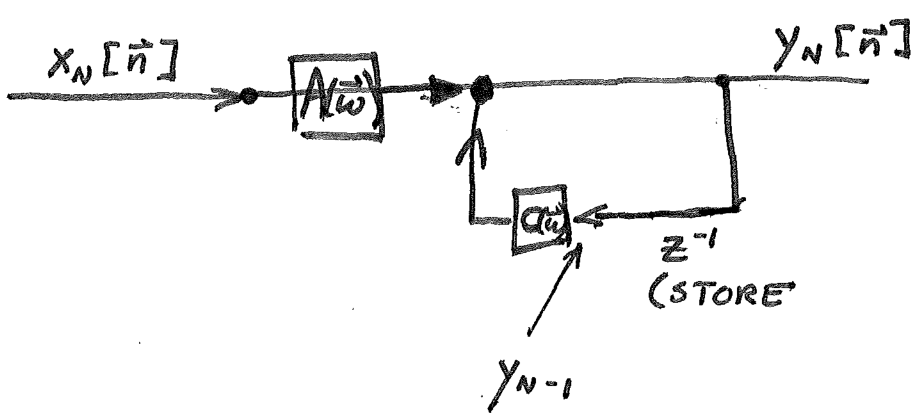
$$y_N[\vec{n}] = a[\vec{n}] * x[\vec{n}] + c[\vec{n}] * y_{N-1}[\vec{n}]$$

$$y_{-1} = 0 \qquad y_0 = a\,x$$

Define

$$x_N[\vec{n}] := x[\vec{n}]\,u[N]$$

$$\Rightarrow y_N[\vec{n}] = a[\vec{n}] * x_N[\vec{n}] + c[\vec{n}] * y_{N-1}[\vec{n}]$$



Resulting Error:

$$\Psi_N = A \sum_{n=0}^{N} C^n \Phi$$

$$S_N = \sum_{n=0}^{N} C^n = 1 + C + \ldots + C^N$$

$$C S_N = C + \ldots + C^N + C^{N+1}$$

$$S_N = \frac{1 - C^{N+1}}{1 - C}$$

Thus $\Psi_N = A \dfrac{1 - C^{N+1}}{1 - C} \Phi$

$$= \Psi \left[1 - C^{N+1} \right]$$

$$E(\vec{\omega}) = \text{Error} = \left| \frac{\Psi_N}{\Psi} - 1 \right| = |C(\vec{\omega})|^{N+1}$$

We know C.

If truncated After N iterations

$$H_N(\vec{\omega}) = A(\vec{\omega}) \sum_{n=0}^{N} C^n(\vec{\omega})$$

$$= A \frac{1 + C^{N+1}}{1 - C}$$

$$= H(\omega) \left[1 \mp C^{N+1} \right]$$

5.2.2. GENERALIZATION OF THE ITERATIVE IMPLEMENTATION

Restriction: $|C| < 1$

(A) Define

$$H(\vec{\omega}) = \frac{A(\vec{\omega})}{B(\vec{\omega})} = \frac{\lambda A(\vec{\omega})}{\lambda B(\vec{\omega})}$$

Redefine

$$C(\vec{\omega}) = 1 - \lambda B(\vec{\omega})$$

Iterative algorithm becomes:

$$\underline{Y}_N = \lambda A \underline{X} + C \underline{Y}_{N-1}$$

Still requires $|C| < 1$, but we now have free parameter, λ.

Ex Let $B > 0$. If $\lambda > 0$

$$\Rightarrow C = 1 - \lambda B < 1$$

Must also have $C > -1$. Choose

$$0 < \lambda < \frac{2}{\max B(\vec{\omega})}$$

$$\Rightarrow |C| < 1$$

(B) More general case:

$B(\vec{\omega})$ complex, <u>but</u> $B(\vec{\omega}) \neq 0$

$$H(\vec{\omega}) = \frac{A(\vec{\omega})}{B(\vec{\omega})} = \frac{\lambda B^*(\vec{\omega}) A(\vec{\omega})}{\lambda |B(\vec{\omega})|^2}$$

Redefine:

$$C(\vec{\omega}) = 1 - \lambda |B(\vec{\omega})|^2$$

Iteration Becomes:

$$\Psi_N(\vec{\omega}) = \lambda B^*(\vec{\omega}) A(\vec{\omega}) \Xi(\vec{\omega}) + C(\vec{\omega}) \Psi_{N-1}(\vec{\omega}) \quad (*)$$

Choose

$$0 < \lambda < \frac{2}{\max |B(\vec{\omega})|^2}$$

Observation: Here, the phase converges in one iteration.

$$\angle \Psi(\vec{\omega}) = \angle \Psi_N(\vec{\omega}) \quad \text{for } i \geq 0$$

$$\Psi_{-1}(\vec{\omega}) = 0$$

Proof: from (*)

$$\angle \Psi_0(\vec{\omega}) = \angle A(\vec{\omega}) + \angle B^*(\vec{\omega}) + \angle \Xi(\vec{\omega})$$

$$= \angle A(\vec{\omega}) - \angle B(\vec{\omega}) + \angle \Xi(\vec{\omega})$$

But $\Psi = \frac{A}{B} \Xi \Rightarrow \angle \Psi = \angle A - \angle B + \angle \Xi$

Thus:

$$\angle \Psi_0(\vec{\omega}) = \angle \Psi(\vec{\omega})$$

further iteration improves $|\cdot|$

Ⓒ ANOTHER GENERALIZATION

$$H(\vec{\omega}) = \frac{\lambda(\vec{\omega})}{\lambda(\vec{\omega})} \frac{A(\vec{\omega})}{B(\vec{\omega})}$$

$$C(\vec{\omega}) = 1 - \lambda(\vec{\omega}) B(\vec{\omega})$$

Iteration:

$$Y_N(\vec{\omega}) = \lambda(\vec{\omega}) A(\vec{\omega}) X(\vec{\omega}) + C(\vec{\omega}) Y_{N-1}(\vec{\omega})$$

$$E_N(\vec{\omega}) = |C(\vec{\omega})|^{N+1}$$

Choose $\lambda(\vec{\omega})$ so that $|C(\vec{\omega})| \simeq 0$ over band of interest.

Optimal: Choose $\lambda = B^{-1} \Rightarrow C = 0$
Converge in one iteration.
B^{-1} can't be implemented directly

Choose λ using FIR filter.

Closer to zero over bigger interval $\Rightarrow$ more computations per iteration.

1. Sequences ~~of~~ with finite support

$$X_z(z_1, z_2) = \sum_{n_1=N_1}^{M_2} \sum_{n_2=N_2}^{M_2} x[n_1, n_2] z_1^{-n_1} z_2^{-n_2}$$

Converges everywhere, but $z_1 = 0$ & $z_2 = 0$

2. Sequences with quadrant support

$$X_z(z_1, z_2) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} x[n_1, n_2] z_1^{-n_1} z_2^{-n_2}$$

If convergent for (z_{01}, z_{02}), then it is convergent for all

$$|z_1| > |z_{01}|, \quad |z_2| \geq |z_{02}|$$

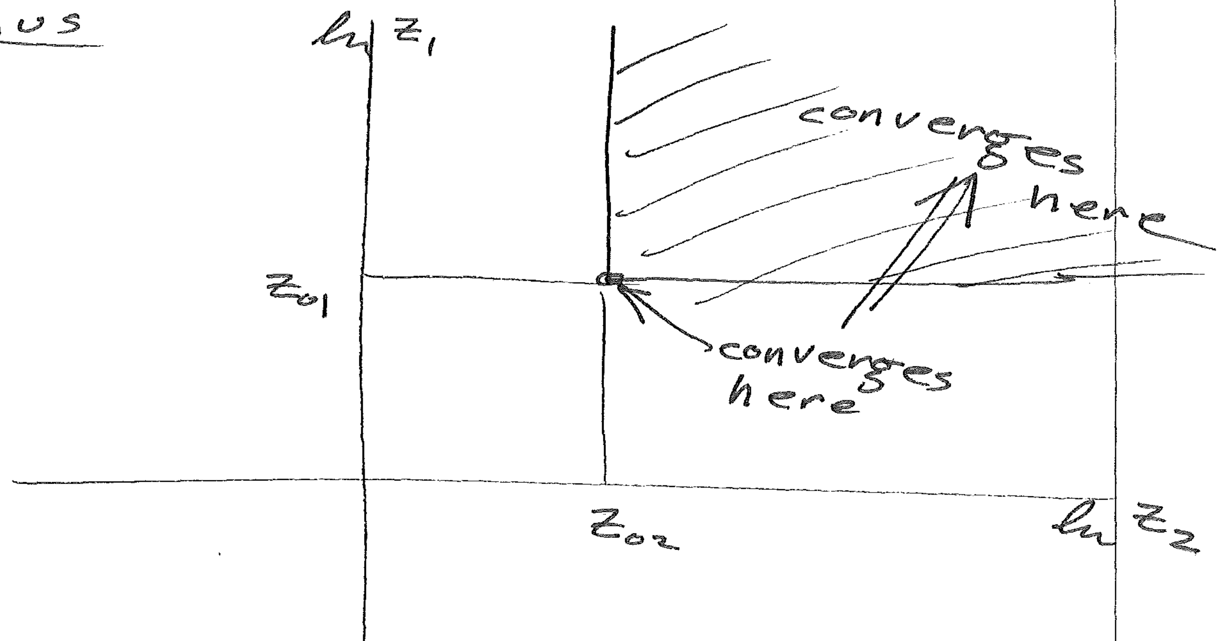
Proof:

$$\infty < \sum_{\substack{n_1=0 \\ n_2}}^{\infty} x[n_1, n_2] z_{01}^{-n_1} z_{02}^{-n_2}$$

$$< \sum_{n_1 n_2=0}^{\infty} x[n_1, n_2] z_1^{-n_1} z_2^{-n_2} \text{ if}$$

since $|z_1|^{-n_1} < |z_{01}|^{-n_2}$ etc.

Thus



6. PROCESSING SIGNALS CARRIED BY PROPAGATING WAVES

6.1. SPACE-TIME SIGNAL

$$s(\vec{x},t) = s(x,y,z;t)$$

SPECTRUM:

$$S(\vec{k},\omega) = \int_{\vec{x}}\int_{t} s(\vec{x},t)\, e^{-j(\omega t - \vec{k}^T\vec{x})}\, d\vec{x}\,dt$$

$$\vec{k} = \underbrace{(k_x, k_y, k_z)}_{\text{Wavenumber}}{}^T$$

$$s(\vec{x},t) = \frac{1}{(2\pi)^4}\int_{k}\int_{\omega} S(\vec{k},\omega)\, e^{j(\omega t - \vec{k}^T\vec{w})}\, d\vec{k}\,d\omega$$

6.1.1. Elemental Signals

$$e(\vec{x},t) = e^{j(\omega_0 t - \vec{k}_0^T\vec{x})}$$

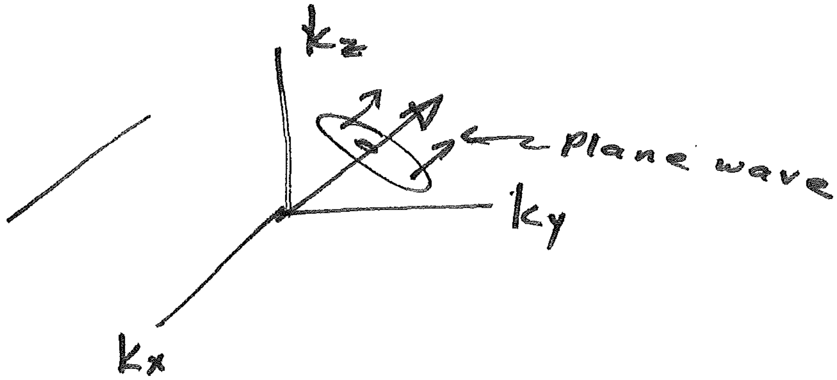
$$\cancel{= e^{j\omega_0(t - \vec{\alpha}_0^T\vec{x})} ; \vec{\alpha}_0 = \vec{k}_0/\omega_0}$$

$$= e^{j\omega_0(t - \vec{\alpha}_0^T\vec{x})} ; \vec{\alpha}_0 = \vec{k}_0/\omega_0 = \text{slowness vector since } \frac{1}{|\vec{\alpha}_0|} = c$$

Then

$$E(\vec{k},\omega) = \delta(\vec{k} - \vec{k}_0)\,\delta(\omega - \omega_0)$$

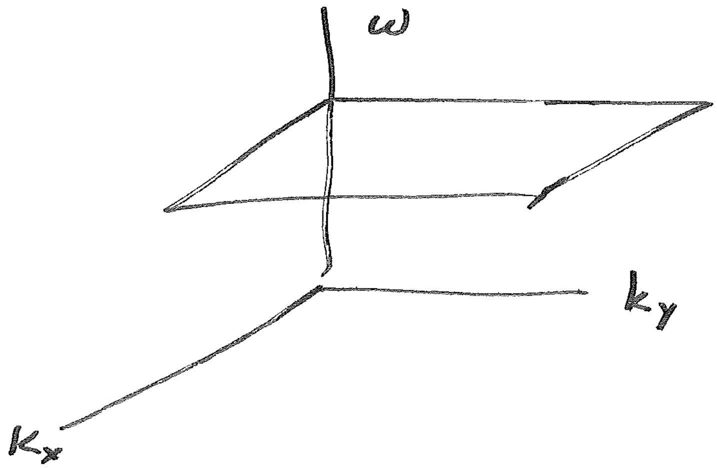
Each point in $\vec{k},\omega$ space corresponds to a plane wave with ~~frequen~~ propagation direction $\vec{k}$ & frequency ω



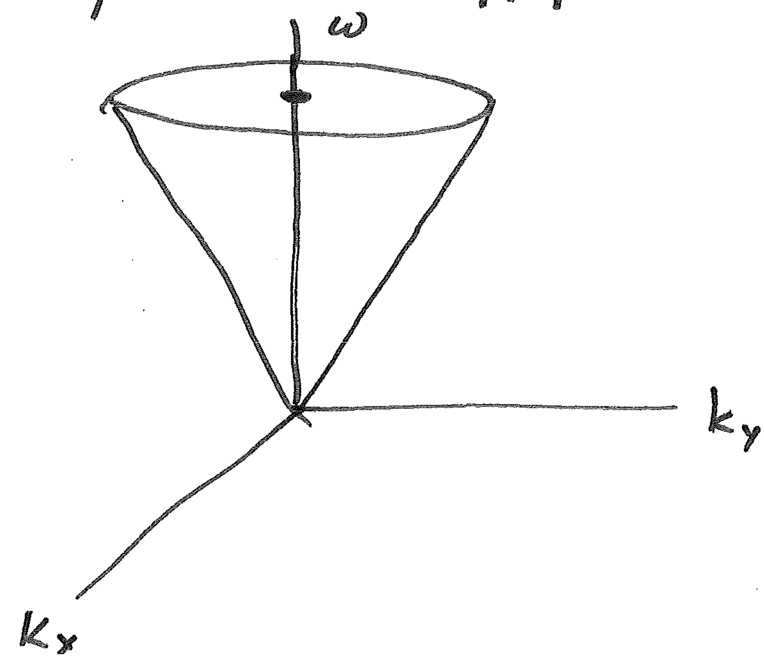
Alternate: Set phase in $e(\vec{x},t)$ = equal to a constant

Assume $k_z = 0$

Same "color" : $\omega = \text{const}$



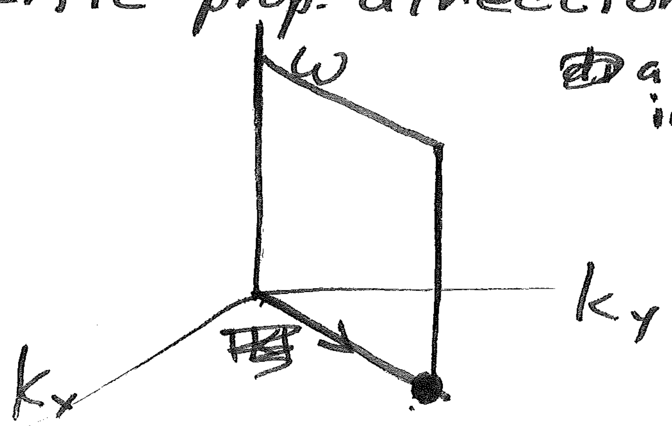
Same speed: $c = \frac{\omega}{|\vec{k}|}$



Constant c, same $\omega \Rightarrow$ intersection

Specific prop. direction: ~~$|\vec{k}| = \text{const}$~~

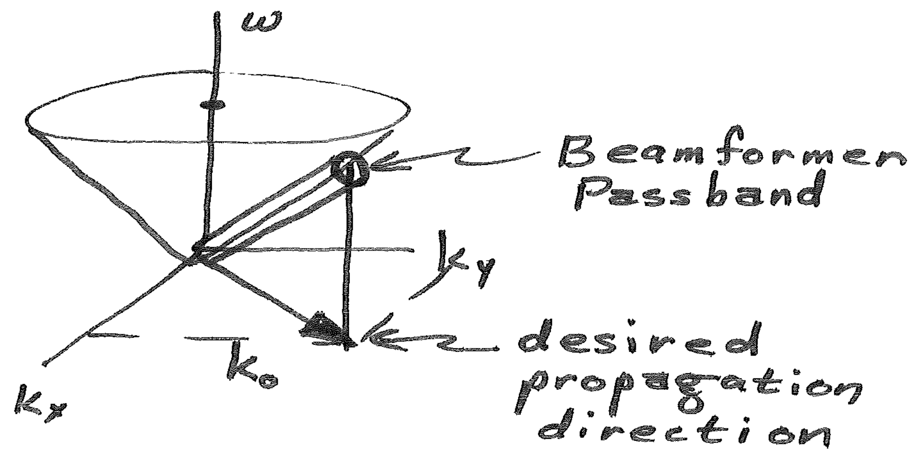
all direction in that of $\vec{k}_0$



6.2. BEAMFORMING

Object: Isolate signal components propagating in particular direction (elaborate (stars))

Assume $c =$ constant



6.2.1. Weighted Delay-and-Sum Beamformer

Assume array of N receivers distributed in space at:

$$\{\vec{x}_i \mid i = 0, 1, \ldots, N-1\}$$

Received signals are samples of $s(\vec{x}, t)$:

$$r_i(t) = s(\vec{x}_i, t)$$

Form "weighted delay-and-sum beamformer"

$$bf(t) = \frac{1}{N} \sum_{i=0}^{N-1} w_i \, r_i(t - \tau_i)$$

Generally choose $\tau_i = -\vec{\alpha}_0 \vec{x}_i$ to get beam in $\vec{\alpha}_0$ direction

(Beam steering)

6.2.2. Array Pattern

Assume: Beam steered to $\vec{\alpha}_o$
(Can't detect exactly)

Incident Plane Wave:

$$s(\vec{x}, t) = e^{j\omega(t - \vec{\alpha}^T \vec{x})}$$

Then

$$bf(t) = \frac{1}{N} \sum_{i=0}^{N-1} w_i \, r_i(t - \tau_i)$$

$$r_i(t) = e^{j\omega(t - \vec{\alpha}^T \vec{x}_i)}$$

$$bf(t) = \frac{1}{N} \sum_{i=1}^{N-1} w_i \, e^{j\omega(t - \tau_i - \vec{\alpha}^T \vec{x}_i)}$$

$$\tau_i = -\vec{\alpha}_o \vec{x}_i$$

$$bf(t) = \frac{1}{N} \sum_{i=1}^{N-1} w_i \, e^{j\omega(t + \vec{\alpha}_o^T \vec{x}_i - \vec{\alpha}^T x_i)}$$

$$= \cancel{\frac{1}{N} \sum_{i=1}^{N-1} w_i \, e^{j\omega(\vec{\alpha}_o \vec{x}_i - \vec{\alpha}^T x_i)}}$$

$$= \left[\frac{1}{N} \sum_{i=1}^{N-1} w_i \, e^{-j\omega(\vec{\alpha} - \vec{\alpha}_o)^T \vec{x}_i} \right] e^{j\omega t}$$

Define "array pattern"

$$W(\vec{k}) = \frac{1}{N} \sum_{i=1}^{N-1} w_i \, e^{-j \vec{k}^T \vec{x}_i}$$

(function of position & weight)

Then

$$bf(t) = W(\omega(\vec{\alpha} - \vec{\alpha}_o)) \, e^{j\omega t}$$

If ω is same

$$bf(t) = W((\vec{k} - \vec{k}_o)) \, e^{j\omega t}$$

$W(\vec{k} - \vec{k}_o)$ is the attenuation suffered by planewave with prop. $\vec{\alpha}$ when ~~to~~ array is tuned to $\vec{\alpha}_o$. Idealy $W(\vec{k} - \vec{k}_o) = C\,\delta(\vec{k} - \vec{k}_o)$

We can decompose any $s(\vec{x},t)$ into a superposition of planewaves:

$$s(\vec{x},t)=\frac{1}{(2\pi)^4}\int_{\vec{k}}\int_{\omega} S(\vec{k},\omega)\, e^{j(\omega t-\vec{k}^T\vec{x})}\, d\vec{k}\, d\omega$$

Then

$$bf(t)=\frac{1}{N}\sum_{i=0}^{N-1} w_i\, r_i(t-\tau_i)$$

$$=\frac{1}{N}\sum_{i=0}^{N-1} w_i \frac{1}{(2\pi)^4}\iint S(\vec{k},\omega)\, e^{-j(\vec{k}-\omega\vec{\alpha}_0)^T\vec{x}_i}\, e^{j\omega t}\, d\vec{k}\, d\omega$$

$$=\frac{1}{(2\pi)^4}\iint S(\vec{k},\omega) \left[\frac{1}{N}\sum_{i=0}^{N-1} w_i\, e^{-j(\vec{k}-\omega\vec{\alpha}_0)\vec{x}_i}\right] e^{j\omega t}\, d\vec{k}\, d\omega$$

$$=\frac{1}{(2\pi)^4}\iint \underbrace{S(\vec{k},\omega)}_{\text{Plane Waves}}\, W\underbrace{(\vec{k}\cdot\omega\vec{\alpha}_0)}_{\text{Corresponding Attenuation}}\, e^{j\omega t}\, d\vec{k}\, d\omega$$

SPECIAL CASE:

$$s(\vec{x},t) = v(t - \vec{\alpha}^T \vec{x}) \Leftarrow$$ All components in the same direction

$$S(\vec{k},t) = \mathcal{F}_{\vec{x}} \mathcal{F}_t \, v(t - \vec{\alpha}^T \vec{x})$$

$$= \mathcal{F}_{\vec{x}} \, V(\omega) e^{+j\vec{\alpha}^T\vec{x}}$$

$$= V(\omega) \, \delta(\vec{k} - \omega\vec{\alpha})$$

Then

$$bf(t) = \frac{1}{(2\pi)^4} \iint_{-\infty}^{\infty} S(\vec{k},\omega) \, W(\vec{k} - \omega\vec{\alpha}_0) \, e^{j\omega t} d\vec{k} d\omega$$

$$= \frac{1}{(2\pi)^4} \iint V(\omega) \, \delta(\vec{k} - \omega\vec{\alpha}) \, W(\vec{k} - \omega\vec{\alpha}_0) e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} V(\omega) \, W\big(\omega(\vec{\alpha} - \vec{\alpha}_0)\big) e^{j\omega t} d\omega$$

If $\vec{\alpha} = \vec{\alpha}_0$:

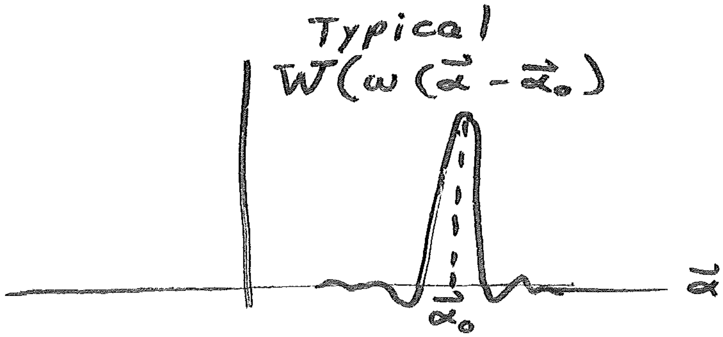
$$bf(t) = \frac{1}{2\pi} W(0) \int_{-\infty}^{\infty} V(\omega) e^{j\omega t} d\omega$$

$$= W(0) \, v(t)$$

Beamformer does not distort signal!

If $\vec{\alpha} \neq \vec{\alpha}_0$, $W(\omega(\vec{\alpha} - \vec{\alpha}_0))$'s argument grows linearly

$\therefore$ Higher ω's will be attenuated more than lower



~~CONVOLUTION INTERPRETATION~~

INTERPRETATION AS FILTER

$s(\vec{x},t)$ = input

$f(\vec{x},t)$ = output

$$f(\vec{x},t) = \iint h(\vec{x}-\vec{\xi}, t-\tau)\, s(\vec{\xi},\tau)\, d\vec{\xi}\, d\tau$$

$$= \frac{1}{(2\pi)^4}\iint H(\vec{k},\omega)\, S(\vec{k},\omega)\, e^{j(\omega t - \vec{k}^T\vec{x})}\, d\vec{k}\, d\omega$$

~~Define $bf(t) = f(\vec{0},t)$~~

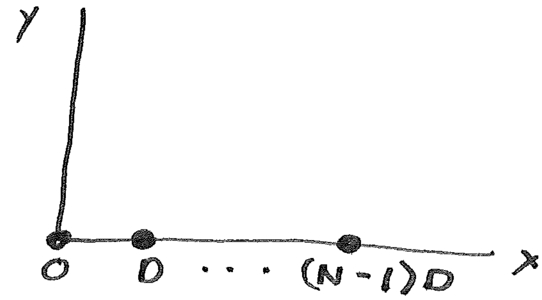
General beamforming:

$$bf(t) = \frac{1}{(2\pi)^4}\iint S(\vec{k},\omega)\, W(\vec{k}-\omega\vec{\alpha}_0)\, e^{j\omega t}\, d\vec{k}\, d\omega$$

Define $bf(t) = f(\vec{0},t)$. Then

$$H(\vec{k},\omega) = \overline{W}(\vec{k}-\omega\vec{\alpha}_0)$$

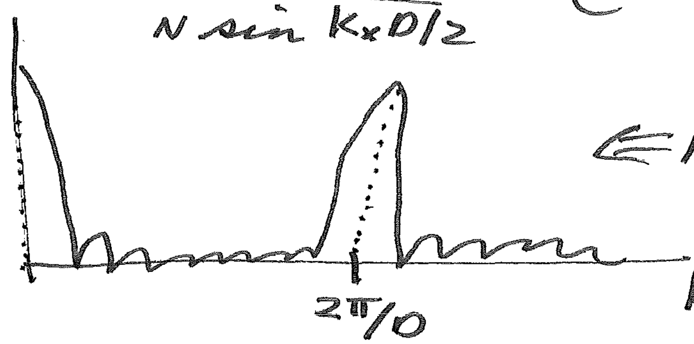
6.2.3. Example of an Array Pattern



$$W(\vec{k}) = \frac{1}{N}\sum_{i=0}^{N-1} e^{-j\vec{k}^T\vec{x}_i}$$

$$= \frac{1}{N}\sum_{i=0}^{N-1} e^{-jk_x iD}$$

$$= \frac{\sin Nk_xD/2}{N\sin k_xD/2}\, e^{-j(N-1)k_xD/2}$$



⇐ Periodic

(May LPF received to avoid lobes ~~avoid~~ avoid spatial aliasing : $D \le \frac{\lambda}{2}$)

To steer beam to $\vec{\alpha}_0$, choose

$$\tau_i = \vec{\alpha}_0^{\,T}\vec{x}_i = -\alpha_{0x}\, iD$$

α_{0x} = x component of $\vec{\alpha}_0$

~~Avoid spati~~

6.2.4. Effect of the Receiver Wave FUNCTION

Want array pattern with:

1. Low Side Lobes
2. ~~Small~~ Big zero order lobes

Same as window!

Indeed, for 1-D array, same as 1-D window.

For 2-D, can choose:

1. outer product in 3-D
2. rotated " "
3. rotated spectrum " "

$$F_L(p) = \int_{-\infty}^{\infty} f(t)e^{-pt}\,dt \qquad f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F_L(p)e^{pt}\,dp$$

$$F_M(s) = \int_0^{\infty} f(x)x^{s-1}\,dx \qquad f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F_M(s)x^{-s}\,ds$$

$$f(n) = \frac{1}{2\pi i}\int_\Gamma F(z)z^{n-1}\,dz \qquad F(z) = \int_{-\infty}^{\infty} \phi(t)z^{-t}\,dt$$

$$= \sum_0^{\infty} f(n)z^{-n}$$

It is clear that the z transform is like the inverse Mellin transform except that t must assume real values whereas s may be complex, and conversely, x is real whereas z may be complex. The contour Γ on the z plane may be understood as follows. It must enclose the poles of the integrand. If the contour $c - i\infty$ to $c + i\infty$ for inverting the Laplace transformation is chosen to the right of all poles, then the circle into which it is transformed by the transformation $z = \exp(-p)$ will enclose all poles. In the common case where $c = 0$ is suitable (all poles of $F_L(p)$ in the left half-plane), the contour Γ becomes the circle $|z| = 1$.

The Abel transform

As soon as one goes beyond the one-dimensional applications of Fourier transforms and into optical-image formation, television-raster display, mapping by radar or passive detection, and so on, one encounters phenomena which invite the use of the Abel transform for their neatest treatment. These phenomena arise when circularly symmetrical distributions in two dimensions are projected in one dimension. A typical example is the electrical response of a television camera as it scans across a narrow line; another is the electrical response of a microdensitometer whose slit scans over a circularly symmetrical density distribution on a photographic plate.

Fractional-order derivatives are also closely connected with the Abel transform, which therefore also arises in fields, such as conduction of heat in solids or transmission of electrical signals through cables, where fractional-order derivatives are encountered.

The Abel transform $f_A(x)$ of the function $f(r)$ is commonly defined as

$$f_A(x) = 2\int_x^{\infty} \frac{f(r)r\,dr}{(r^2 - x^2)^{\frac{1}{2}}}.$$

The choice of the symbols x and r is suggested by the many applications in which they represent an abscissa and a radius, respectively, in the same plane.

The above formula may be written EE 521

$$f_A(x) = \int_0^{\infty} k(r,x)f(r)\,dr,$$

where
$$k(r,x) = \begin{cases} 2r(r^2 - x^2)^{-\frac{1}{2}} & r > x \\ 0 & r < x. \end{cases}$$

The kernel $k(r,x)$, regarded as a function of r in which x is a parameter, shifts to the right as x increases, and it also changes its form. A slight change of variable leads to a kernel which simply shifts without change of form. Thus putting $\xi = x^2$ and $\rho = r^2$, and letting $f_A(x) = F_A(x^2)$ and $f(r) = F(r^2)$, we have

$$F_A(\xi) = \int_0^{\infty} K(\xi - \rho)F(\rho)\,d\rho,$$

where
$$K(\xi) = \begin{cases} (-\xi)^{-\frac{1}{2}} & \xi < 0 \\ 0 & \xi \geqslant 0; \end{cases}$$

alternatively,
$$F_A(\xi) = \int_\rho^{\infty} \frac{F(\rho)\,d\rho}{(\rho - \xi)^{\frac{1}{2}}},$$

or again,
$$F_A = K * F.$$

When necessary, F_A will be referred to as the "modified Abel transform of F." Having reduced the formula to a convolution integral, we may take Fourier transforms and write

$$\bar{F}_A = \bar{K}\bar{F}.$$

Since
$$\bar{K}(s) = \frac{1}{(-2is)^{\frac{1}{2}}},$$

if follows that
$$\bar{F} = (-2is)^{\frac{1}{2}}\bar{F}_A$$
$$= -\frac{1}{\pi}\frac{1}{(-2is)^{\frac{1}{2}}}\,i2\pi s\bar{F}_A$$

whence
$$F = -\frac{1}{\pi}K * F'_A;$$

that is,
$$F(\rho) = -\frac{1}{\pi}\int_\rho^{\infty} \frac{F'_A(\xi)\,d\xi}{(\xi - \rho)^{\frac{1}{2}}},$$

The solution of the modified Abel integral equation enables F to be expressed in terms of the derivative of F_A. Integrating the solution by parts, or choosing different factors for the transform of F, we obtain a solution in terms of the second derivative of F_A:

$$F = \frac{2}{\pi}\mathcal{K} * F''_A,$$

where
$$\mathcal{K}(\xi) = \begin{cases} (-\xi)^{\frac{1}{2}} & \xi < 0 \\ 0 & \xi \geqslant 0. \end{cases}$$

Table 12.9 Some Abel transforms

$f(r)$		$f_A(x)$	
$\Pi(r/2a)$	Disk	$2(a^2 - x^2)^{\frac{1}{2}}\Pi(x/2a)$	Semiellipse
$(a^2 - r^2)^{-\frac{1}{2}}\Pi(r/2a)$		$\pi\Pi(x/2a)$	Rectangle
$(a^2 - r^2)^{\frac{1}{2}}\Pi(r/2a)$	Hemisphere	$\frac{1}{2}\pi(a^2 - x^2)\Pi(x/2a)$	Parabola
$(a^2 - r^2)\Pi(r/2a)$	Paraboloid	$\frac{4}{3}(a^2 - x^2)^{\frac{3}{2}}\Pi(x/2a)$	
$(a^2 - r^2)^{\frac{3}{2}}\Pi(r/2a)$		$(3\pi/8)(a^2 - x^2)^2\Pi(x/2a)$	
$a\Lambda(r/a)$	Cone	$[a(a^2 - x^2)^{\frac{1}{2}} - x^2 \cosh^{-1}(a/x)]\Pi(x/2a)$	
$\pi^{-1}\cosh^{-1}(a/r)\Pi(r/2a)$		$a\Lambda(x/a)$	Triangle
$\delta(r - a)$	Ring impulse	$2a(a^2 - x^2)^{-\frac{1}{2}}\Pi(x/2a)$	
$\exp(-r^2/2\sigma^2)$	Gaussian	$(2\pi)^{\frac{1}{2}}\sigma \exp(-x^2/2\sigma^2)$	Gaussian
$r^2 \exp(-r^2/2\sigma^2)$		$(2\pi)^{\frac{1}{2}}\sigma(x^2 + \sigma^2)\exp(-x^2/2\sigma^2)$	
$(r^2 - \sigma^2)\exp(-r^2/2\sigma^2)$		$(2\pi)^{\frac{1}{2}}\sigma x^2 \exp(-x^2/2\sigma^2)$	
$(a^2 + r^2)^{-1}$		$\pi(a^2 + x^2)^{-\frac{1}{2}}$	
$J_0(2\pi ar)$		$(\pi a)^{-1}\cos 2\pi ax$	
$2\pi\left[r^{-3}\int_0^r J_0(r)\,dr - r^{-2}J_0(r)\right] = M(r)$		$\operatorname{sinc}^2 x$	
$\delta(r)/\pi\lvert r\rvert$		$\delta(x)$	
$2a \operatorname{sinc} 2ar$		$J_0(2\pi ax)$	
$\pi^{-1}ar^{-1}J_1(2\pi ar)$		$\operatorname{sinc} 2ax$	

Since $\bar{K}$ is nowhere zero, the solution is unique (except for additive null functions).

Reverting to f and f_A, we may write the solutions as

$$f(r) = -\frac{1}{\pi}\int_r^\infty \frac{f'_A(x)\,dx}{(x^2 - r^2)^{\frac{1}{2}}} = -\frac{1}{\pi}\int_r^\infty (x^2 - r^2)^{\frac{1}{2}} \frac{d}{dx}\left[\frac{f'_A(x)}{x}\right] dx,$$

or, if the integral is zero beyond $x = r_0$, and allowing for the possibility that the integrand may behave impulsively at r_0, we have

$$\begin{aligned} f(r) &= -\frac{1}{\pi}\int_r^{r_0} \frac{f'_A(x)\,dx}{(x^2 - r^2)^{\frac{1}{2}}} + \frac{f_A(r_0)}{\pi(r_0{}^2 - r^2)^{\frac{1}{2}}} \\ &= -\frac{1}{\pi}\int_r^{r_0} (x^2 - r^2)^{\frac{1}{2}} \frac{d}{dx}\left[\frac{f'_A(x)}{x}\right] dx - \frac{f'_A(r_0)}{\pi r_0}(r_0{}^2 - r^2)^{\frac{1}{2}}. \end{aligned}$$

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Useful relations for checking Abel transforms are

$$\int_{-\infty}^{\infty} f_A(x)\,dx = 2\pi\int_0^\infty f(r)r\,dr$$

and

$$f_A(0) = 2\int_0^\infty f(r)\,dr.$$

Another property is that

$$K * K * F' = -\pi F;$$

that is, the operation $K\,*$ applied twice in succession annuls differentiation; then F_A is the half-order integral of F, and conversely, F is the half-order differential coefficient of F_A. To prove this, note that if $F_A = K * F$ implies that $F = -\pi^{-1}K * F'_A$, then it follows further that $F'_A = K * F'$; whence

$$K * K * F' = K * F'_A = -\pi F.$$

In Table 12.9 the first eight examples are to be taken as zero for r and x greater than a.

Numerical evaluation of Abel transforms is comparatively simple in view of the possibility of conversion to a convolution integral. One first makes the change of variable, then evaluates sums of products of $K(\rho)$ and $f(\xi - \rho)$ at discrete intervals of ρ. The values of K turn out to be the same, however fine an interval is chosen, save for a normalizing factor; consequently, a universal table of values (see Table 12.10) can be set up for permanent reference. The table shows coefficients for immediate use with values of F read off at $\rho = \frac{1}{2}, 1\frac{1}{2}, \ldots, 9\frac{1}{2}$, the scale of ρ being such that F becomes zero or negligible at $\rho = 10$. The table gives mean values of K over the intervals $0 - 1, 1 - 2, \ldots$. Thus at $\rho = n + \frac{1}{2}$ the value is

$$\int_n^{n+1} K(-\rho)\,d\rho = 2(n+1)^{\frac{1}{2}} - 2n^{\frac{1}{2}}.$$

Table 12.10 Coefficients for performing or inverting the Abel transformation

ρ	K	ρ	K	ρ	K	ρ	K
$\frac{1}{2}$	2.000	$5\frac{1}{2}$	0.427	$10\frac{1}{2}$	0.309	$15\frac{1}{2}$	0.254
$1\frac{1}{2}$	0.828	$6\frac{1}{2}$	0.393	$11\frac{1}{2}$	0.295	$16\frac{1}{2}$	0.246
$2\frac{1}{2}$	0.636	$7\frac{1}{2}$	0.364	$12\frac{1}{2}$	0.283	$17\frac{1}{2}$	0.239
$3\frac{1}{2}$	0.536	$8\frac{1}{2}$	0.343	$13\frac{1}{2}$	0.272	$18\frac{1}{2}$	0.233
$4\frac{1}{2}$	0.472	$9\frac{1}{2}$	0.325	$14\frac{1}{2}$	0.263	$19\frac{1}{2}$	0.226

Multidimensional Projection Windows

WEN-CHUNG STEWART WU, KWAN F. CHEUNG, AND ROBERT J. MARKS, II

Abstract —A one-dimensional window is chosen from the large catalog of those available primarily due to its leakage-resolution tradeoff (LRT). Is it possible to generalize a 1-D window to higher dimensions such that the

Manuscript received June 6, 1987; revised February 18, 1988. This paper was recommended by Associate Editor D. M. Goodman.

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IEEE Log Number 8822405.

0098-4094/88/0900-1168$01.00 ©1988 IEEE

window's 1-D properties are homogeneously preserved? If we require that the window be continuous and bounded the answer is usually no. Bounded (projection window) generalizations do exist for the Parzen and Tukey–Hanning windows. The resulting windows, however, are very close to that window obtained by simply rotating the 1-D window into two ensions.

Introduction

When choosing from the large catalog of standard 1-D windows [1]–[2], one is largely motivated by the window's leakage-resolution tradeoff (LRT). Is it possible to generalize these windows to two and higher dimensions such that the 1-D window properties are preserved in each 1-D slice? If we require these multidimensional windows to be bounded and continuous, the answer is usually negative. In the two cases considered in this correspondence where bounded 2-D generalizations do exist, the resulting windows are close to those obtained by the rotation generalization of 1-D windows [3].

A short review of the outer product and rotation of 1-D window generalization methods is given in the next section. In both cases, the LRT is altered in the transformation. In order to homogeneously maintain the 1-D window properties, the higher dimension window must be chosen so that its projection onto one dimension results in the 1-D window. Unfortunately, this requires unbounded generalizations in many cases of interest. The Parzen and Tukey–Hanning windows are exceptions. For the discrete case, bounded projection windows can be formed such that desired LRT is preserved inhomogeneously at a number of angular orientations.

Preliminaries

There are an wealth of 1-D windows with various LRT's. A window, $w_1(t)$ has finite extent:

$$w_1(t) = w_1(t)\Pi(t/2\tau)$$

(where $\Pi(t)=1$ for $|t|\leqslant 1/2$ and is zero elsewhere), is normalized with

$$w_1(0)=1$$

and is an even function, i.e.,

$$w_1(t)=w_1(-t).$$

The spectrum of a window is defined by

$$W_2(\omega) = \int_{-\infty}^{\infty} w_1(t)\exp(-j\omega t)\,dt.$$

The area of a window is

$$A = \int_{-\infty}^{\infty} w_1(t)\,dt = W_1(0).$$

The magnitude of a typical window spectrum is shown in Fig. 1. For good resolution, the main lobe width, Δ, should be small, and for minimal spectral leakage, the normalized side lobe magnitude, δ, should also be small. Invariably, however, decreasing one of these parameters increases the other.

A 2-D window $w_2(t_1,t_2)$, with spectrum

$$W_2(\omega_1,\omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w_2(t_1,t_2)\exp[-j(\omega_1 t_1+\omega_2 t_2)]\,dt_1\,dt_2$$

is commonly generated from a 1-D counterpart by either the er product or window rotation techniques [3]. The outer duct window is

$$w_2^{op}(t_1,t_2) = w_1(t_1)\,w_1(t_2)$$

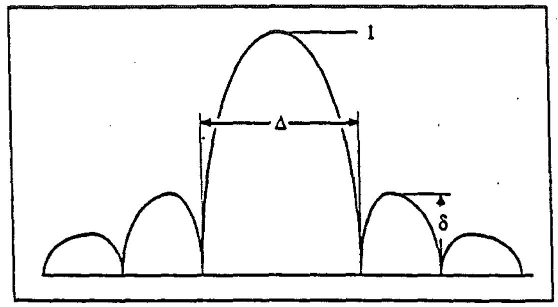

Fig. 1. The normalized spectrum of a typical 1-D window, $|W_1(\omega)|/A$. The values of Δ and δ parameterize the window's resolution and leakage, respectively.

and the rotated window, initially suggested by Huang [4], is

$$w_2^{rw}(t_1,t_2) = w_1\left(\sqrt{t_1^2+t_2^2}\right).$$

In either case, if w_1 is a "good" window, then so is w_2. For certain applications, (e.g., "good" filter design) such dimensional generalizations are acceptable. In other cases, such as spectral estimation, a small perturbation in window shape can significantly alter results [5]. Both the outer product and the rotated window significantly alter the LRT of the corresponding 1-D window.

To illustrate the effects of outer product and rotational dimensional generalization, we choose a boxcar window

$$w_1(t) = \Pi(t/2\tau).$$

It follows that

$$W_1(\omega) = 2\sin(\tau\omega)/\omega$$

for which

$$\Delta = 6.3/\tau;\quad \delta = 0.22. \tag{1}$$

For the outer product window, in general,

$$W_2^{op}(\omega_1,\omega_2) = W_1(\omega_1)\,W_2(\omega_2).$$

The result is a window with an identical LRT as the 1-D window in the t_1 and t_2 directions. Indeed

$$W_2(\omega_1,0) = A\,W_1(\omega_1).$$

However, in other directions, the LRT can be significantly altered. For example, in the (t_1,t_2) plane, the Δ parameter for the window resolution in the $\pm 45°$ directions in $\sqrt{2}$ times that of the 0° and 90° directions. Consider, specially, the boxcar window, for which

$$W_2(\omega_1,\omega_2) = 4\sin(\tau\omega_1)\sin(\tau\omega_2)/(\omega_1\omega_2).$$

The 1-D slice of this window along the 45° diagonal is

$$W_2^{op}\left(\omega_1/\sqrt{2},\omega_2/\sqrt{2}\right) = 4\sin^2\left(\omega/\sqrt{2}\right)/\omega^2$$

which is the spectrum of a Bartlett (triangular) window. The parameters of this window with respect to those in (1) are

$$\Delta_{45°} = \sqrt{2}\,\Delta \cong 8.9/\tau$$

and

$$\delta_{45°} = 0.047 \cong (0.22)^2 = \delta^2.$$

Clearly, the LRT is significantly altered.

For the rotated window, the window spectrum can be written as

$$W_2^{rw}(\omega_1,\omega_2) = W_2(\rho) = 2\pi\int_0^{\infty} r W_1(r) J_0(r\rho)\,dr \tag{2}$$

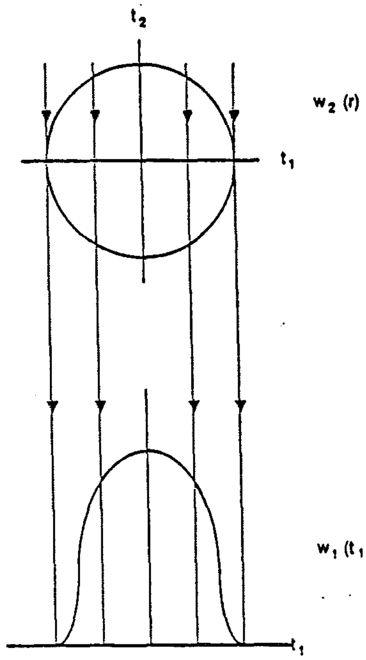

Fig. 2. Illustration of the mechanics of forming a 1-D projection, $w_1(t_1)$, from a 2-D circularly symmetric function $\hat{w}_2(r)$, $(r^2 = t_1^2 + t_2^2)$. If $w_1(t_1)$ is the projection of $w_2(r)$, then $w_2(r)$ homogeneously preserves the LRT of its 1-D counterpart.

where

$$\rho = \sqrt{\omega_1^2 + \omega_2^2}$$

and

$$r = \sqrt{t_1^2 + t_2^2}.$$

Equation (2) is the familiar Hankel transform [6] which results from Fourier transforming a circularly symmetric 2-D function. Although the rotation window does not have the directional inhomogeneity of the outer product window, the LRT of the original window is also significantly altered. Consider the rotated boxcar window with spectrum

$$W_2^{rw}(\rho) = 2\pi\tau J_1(\tau\rho)/\rho.$$

Here

$$\Delta_{rw} \cong 7.7/\tau = 1.2\,\Delta$$

and

$$\delta_{rw} = 0.13 \cong 0.59\,\delta.$$

The Projection or Rotated Spectrum Window

The 2-D window, $w_2^p(r)$, that preserves the LRT of its corresponding 1-D window in all directions will be referred to as the projection or rotated spectrum window. The window can be thought of in one of two equivalent ways:

1) Projection

With reference to Fig. 2, $w_2^p(r)$ is the window whose projection is the 1-D design window,

$$w_1(t_1) = \int_{-\infty}^{\infty} w_2^p(r)\, dt_2. \tag{3}$$

By straightforward manipulation, w_1 is recognized as the Abel transform of w_2^p:

$$w_1(t_1) = 2\int_{t_1}^{\infty} r w_2^p(r)/\sqrt{r^2 - t_1^2}\, dr.$$

Thus the 2-D window can be obtained from an inverse Abel transform [6]:

$$w_2(r) = \frac{1}{\pi}\int_r^{\infty} \sqrt{t_1^2 - r^2}\, \frac{d}{dt_1}\left(\frac{w_1'(t_1)}{t_1}\right) dt_1$$

where the prime denotes differentiation. Since $w_1(t_1)$ is zero for $|t_1| > \tau$, an equivalent expression is [6]:

$$w_2(r) = \frac{1}{\pi}\int_r^{\infty} \sqrt{t_1^2 - r^2}\, \frac{d}{dt_1}\left(\frac{w_1'(t_1)}{t_1}\right) dt_1 - \frac{w_1'(\tau)}{\pi\tau}\sqrt{\tau^2 - r^2}, \quad \text{for } |r| \leqslant \tau. \tag{4}$$

2) Rotated Spectrum

The spectrum of the projection window is the rotation of the spectrum of the 1-D window. That is,

$$W_2^p(\rho) = W_1(\rho).$$

The window can thus be obtained by an inverse Hankel transform:

$$w_2^p(r) = \int_0^{\infty} \rho W_1(\rho) J_0(r\rho)\, d\rho/2\pi.$$

Through this definition of projection window, one can clearly see that the LRT of the original window is preserved in the 2-D generalization in all directions.

The equivalence of this and the projection window follows immediately from the continuous version of the projection-slice theorem [3] or, for even functions, from the equality of an Abel transform to Fourier Transform followed by an inverse Hankel transform [6].

Examples

1) The *Parzen Window* is obtained by convolving two identical (Bartlett type) triangular windows and normalizing. The result is [7]

$$w_1(t_1) = \begin{cases} 1 - 6\left(\dfrac{t_1}{\tau}\right)^2 + 6\left|\dfrac{t_1}{\tau}\right|^3, & |t_1| \leqslant \tau/2 \\ 2\left(1 - \left|\dfrac{t_1}{\tau}\right|\right)^3, & \tau/2 \leqslant |t_1| \leqslant \tau \\ 0, & |t_1| \geqslant \tau. \end{cases}$$

Recognizing that $w_1'(\tau) = 0$, we obtain from (4) after some variable substitution:

$$\hat{w}_2(r) = w_2^p(r\tau) = \begin{cases} \dfrac{9}{\pi}\left(\dfrac{b}{2} - r^2 \ln\left(\dfrac{\frac{1}{2} - b}{r}\right)\right) + \dfrac{6}{\pi}\left(\dfrac{9b}{4} - \dfrac{3a}{2} + c\ln\left(\dfrac{1+a}{\frac{1}{2} + b}\right)\right), & 0 \leqslant r \leqslant \dfrac{1}{2} \\ \dfrac{6}{\pi}\left(\dfrac{-3a}{2} + c\ln\left(\dfrac{1+a}{r}\right)\right), & \dfrac{1}{2} \leqslant r \leqslant 1 \end{cases}$$

A pedagogical $N=5$ closed-form example, taken directly from an Abel transform table [6], is

$$w_1(r_1) = \left[1-\left(\frac{r_1}{\tau}\right)^2\right]\Pi(r_1/2\tau)$$

$$w_2(r_2) = \frac{2}{\pi\tau^2}\left(\tau^2 - r_2^2\right)^{1/2}\Pi(r_2/2\tau)$$

$$w_3(r_3) = \frac{1}{\pi\tau}\Pi(r_3/2\tau)$$

$$w_4(r_4) = \frac{1}{(\pi\tau)^2\left(\tau^2 - r_4^2\right)^{1/2}}\Pi(r_4/2\tau)$$

$$w_5(r_5) = \frac{2}{\pi^2\tau}\delta(r_5 - \tau)$$

where δ is the unit impulse function.

An alternate approach to multidimensional projection windows follows from the property that the inverse Hankel transform of a Fourier transform is equivalent to an Abel transform. Thus, the $(N-1)$ inverse Abel transform can be performed in the Fourier domain. Bracewell [6] has shown that these operations can be condensed into the single transform:

$$w_N(r_N) = \frac{N}{(2\pi r_N)^{N/2}}\int_0^\infty W_1(\omega)\,J_{N/2-1}(\omega r_N)\,\omega^{N/2}\,d\omega$$

where $J_{(N/2)-1}$ is the Bessel function of order $(N/2)-1$.

Conclusions

The projection window preserves the LRT of the 1-D window from which it is designed. This is not in general true for the outer product and rotation window generalizations. The Parzen and Tukey–Hanning windows were shown to have straightforward N-D projectional window equivalents. Many other commonly used windows, however, were shown to have unbounded projection. Further work in the digital equivalent of the dimensional generalization is in order. Here, boundedness need not be an issue.

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[1] A. V. Oppenheim and R. W. Schafer, *Digital Signal Processing.* Englewood Cliffs, NJ: Prentice-Hall, 1975.

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[4] T. S. Huang, "Two dimensional windows," *IEEE Trans. Audio Electroacoust.*, vol. AU-20, pp. 88–89, 1972.

[5] F. J. Harries, "On the use of windows for harmonic analysis with the discrete fourier transform," *Proc. IEEE*, vol. 66, pp. 26–50, 1978.

[6] R. N. Bracewell, *The Fourier Transform and its Applications.* 2nd edition, (revised) New York: McGraw-Hill, 1986.

[7] M. B. Priestley, *Spectral Analysis and Time Series.* New York: Academic, 1981.

[8] W.-C. S. Wu, "Multidimensional Windows Design Using Abel Projection," Master thesis, Univ. Washington, Seattle, 1985.

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Solutions

Final Examination: EE521

Robert J. Marks II

- Do all of your work in this test booklet.
- The test begins promptly at 8:30 AM.
- The test is closed book and closed notes. Each student is allowed two $8\frac{1}{2} \times 11$ sheet of paper with notes. Calculators are allowed.
- Each problem is worth the same number of points.
- After the test, you may forget about this course for the rest of the year.

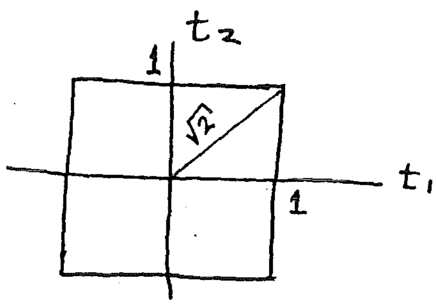
1. The first problem is your work on the McClellan transform. Please attach it to this booklet when you hand in your test.

2. Provide a detailed sketch of the projection of

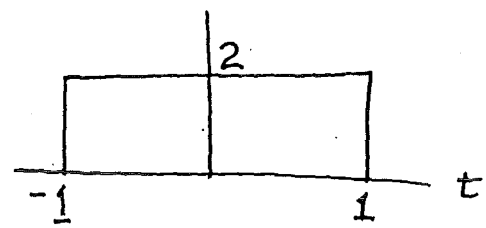
$$x(t_1, t_2) = \Pi\left(\frac{t_1}{2}\right)\left(\frac{t_2}{2}\right)$$

(a) onto the t_2 axis,

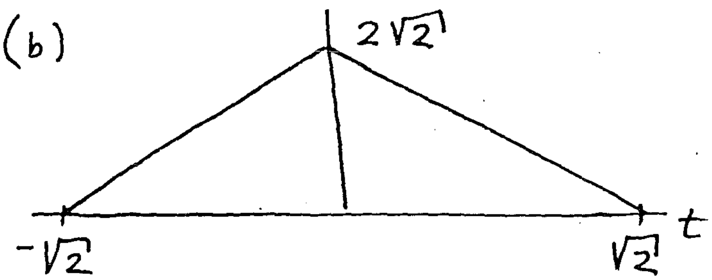
(b) perpendicular to the line $t_1 = t_2$,



(a)



(b)



3. Denote an Abel transform, $f_A(t)$, of a radial function, $f(r)$, by

$$f(r) \leftrightarrow f_A(t).$$

(a) What is the scaling theorem for Abel transforms? In other words,

$$f\left(\frac{r}{M}\right) \leftrightarrow ?$$

You may assume that $M > 0$.

(b) Given the Abel transform pair

$$\Pi(r) \leftrightarrow \left(1 - 4t^2\right)^{\frac{1}{2}} \Pi(t),$$

evaluate the Abel transform of the annulus

$$f(r) = \begin{cases} 1 & ; 1 \leq r \leq 2 \\ 0 & ; \text{otherwise} \end{cases}$$

$$f_A(t) = 2\int_t^\infty \frac{f(r)\, r\, dr}{\sqrt{r^2 - t^2}}$$

(a) $$f\left(\frac{r}{M}\right) \longleftrightarrow 2\int_t^\infty \frac{f(\hat{r}/M)\, \hat{r}\, d\hat{r}}{\sqrt{\hat{r}^2 - t^2}} \underset{r = \frac{\hat{r}}{M}}{=} 2\int_{\frac{t}{M}} \frac{f(r)\,(rM)\,(M\,dr)}{\sqrt{(Mr)^2 - t^2}}$$

$$= 2M \int_{\frac{t}{M}} \frac{f(r)\, r\, dr}{\sqrt{r^2 - \left(\frac{t}{M}\right)^2}} = M\, f_A(t/M)$$

(b) $$f(r) = \Pi\left(\frac{r}{4}\right) - \Pi\left(\frac{r}{2}\right) \longleftrightarrow 4\left(1 - 4\left(\frac{t}{4}\right)^2\right)^{\frac{1}{2}} \Pi\left(\frac{t}{4}\right) - 2\left(1 - 4\left(\frac{t}{2}\right)^2\right)^{\frac{1}{2}} \Pi\left(\frac{t}{2}\right)$$

$M = 4$ $\qquad$ $M = 2$

Simplify if desired.

4. Consider the component filter (transformation function)

$$F(\omega_1, \omega_2) = \cos\left(\frac{\omega_1 - \omega_2}{2}\right).$$

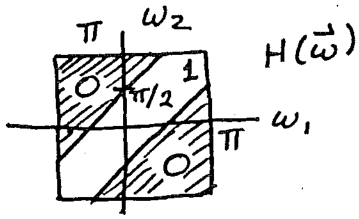
In the $2\pi \times 2\pi$ square in the (ω_1, ω_2) plane, we desire a two dimensional filter

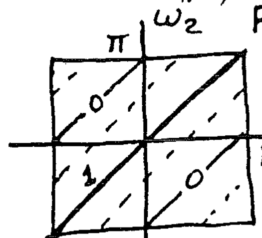
$$H(\omega_1, \omega_2) = \begin{cases} 1 & ; |\omega_1 - \omega_2| \le \frac{\pi}{2} \\ 0 & ; \text{otherwise} \end{cases}$$

Make a **detailed sketch** of the prototype filter

$$H(\omega) = \sum_{n=0}^{N} a_n \cos(n\omega).$$

(Do **not** evaluate any values for the a_n's.)

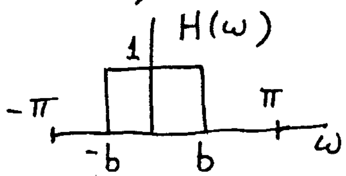




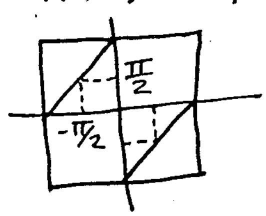
$$H(\omega_1, \omega_2) = \sum_{n=0}^{N} a_n T_n\left[F(\omega_1, \omega_2)\right] = \sum_{n=0}^{N} a_n T_n\left[\cos\left(\frac{\omega_1 - \omega_2}{2}\right)\right]$$

Set $\omega = \omega_1 = -\omega_2 \Rightarrow H(\omega, -\omega) = \sum_{n=0}^{N} a_n T_n\left[\cos\omega\right] = \sum_{n=0}^{N} a_n \cos n\omega = H(\omega)$

Thus, $H(\omega)$ maps to $\omega = \omega_1 = -\omega_2$



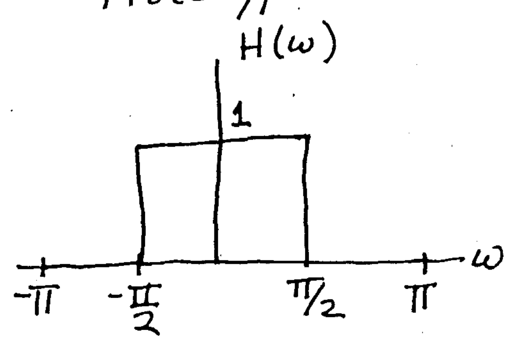
$H(\pm b)$ maps to $(\omega_1, \omega_2) = (\pm b, \mp b)$



Clearly, choose

$b = \pi/2$

Prototype:



Other checks:

$H(0)$ maps to $H(0,0) = 1 \Leftarrow$ GOOD

$H(\pm\pi)$ " " $H(\pm\pi, \mp\pi) = 0 \Leftarrow$ GOOD

5. The IIR filter $H(\omega_1, \omega_2)$ is iteratively implemented where

$$B(\omega_1, \omega_2) = \frac{1}{H(\omega_1, \omega_2)} = 1 - \frac{1}{2}\cos^2(\omega_1)\cos^2(\omega_2).$$

Evaluate the required number of iterations, I, required to assure the maximum error of both the output and the corresponding transfer function does not exceed $\frac{1}{256}$.

$$C = 1 - B = \frac{1}{2}\cos^2\omega_1 \cos^2\omega_2$$

$$E_I = |C|^{I+1}$$

$$(E_I)_{max} = |C|_{MAX}^{I+1}$$

$$|C|_{max} = \frac{1}{2}$$

$$\therefore (E_I)_{max} = \left(\frac{1}{2}\right)^{I+1} = \frac{1}{256} = \left(\frac{1}{2}\right)^8$$

$$\Rightarrow I = 7 \text{ iterations}$$

University of Washington Correspondence

INTERDEPARTMENTAL

Stewart Wu,

1. Attached is a copy of the EE595 exam and solutions ~~to~~ prob 1. You should have access to the other 3.

Please ~~make~~ write up. the solus

2. Since you are leaving on the 14th, I have made the exams due on the 10th @ 1 P.M. At this time, please collect them from ~~to~~ me and grade them. We should talk about how to compute the final grade.

3. I will be giving the 1 page written summaries to you on Wed @ 4:30. Please grade them for clarity. Even though you (nor I) will understand everything, do the best you can. I envision ~~all~~ almost everyone getting a 9 or 10 out of 10 possible points.

Bob Marks

EE595 Test solutions

1.a. $$\mathcal{X}(j\omega_1, j\omega_2) = \int_0^\infty \int_0^\infty x(t_1,t_2)\, t_1^{j\omega_1-1} t_2^{j\omega_2-1}\, dt_1 dt_2$$

$$\mathcal{M}\, x\left(\frac{t_1}{A}, \frac{t_2}{B}\right) = \int_0^\infty \int_0^\infty x\left(\frac{t_1}{A}, \frac{t_2}{B}\right) t_1^{j\omega_1-1} t_2^{j\omega_2-1}\, dt_1 dt_2$$

$$\tau_1 = t_1/A \quad , \quad \tau_2 = t_2/B$$

$$\mathcal{M}\, x\left(\frac{t_1}{A}, \frac{t_2}{B}\right) = \int_0^\infty\int_0^\infty x(\tau_1 \tau_2)\, (\tau_1 A)^{j\omega-1} (\tau_2 B)^{j\omega_2-1}$$
$$\times A\, d\tau_1\, B\, d\tau_2$$

But $|A^{j\omega}| = |e^{j(\ln A)\omega}| = 1$. Thus:

$$\left|\mathcal{M}\; x\left(\frac{t_1}{A}, \frac{t_2}{B}\right)\right| = \left| \int_0\int x(\tau_1\tau_2)\, \tau_1^{j\omega_1} \tau_2^{j\omega_2}\, d\tau_1 d\tau_2 \right|$$
$$= |\mathcal{M}\, x(t_1, t_2)|$$

b. $$y(t_1, t_2) = \int\int_0^\infty x(\tau_1, \tau_2)\, h(t_1\tau_1, t_2\tau_2)\, d\tau_1 d\tau_2$$

$$\mathcal{Y}(s_1, s_2) = \int_0^\infty\int_0^\infty y(t_1, t_2)\, t_1^{s_1-1} t_2^{s_2-1}\, dt_1 dt_2$$
$$= \int_{\tau_1=0}^\infty \int_{\tau_2=0}^\infty x(\tau_1 \tau_2) \left[\int_{t_1=0}^\infty \int_{t_2=0}^\infty h(t_1\tau_1, t_2\tau_2) \right.$$
$$\left. t_1^{s_1-1} t_2^{s_2-1}\, dt_1 dt_2 \right] d\tau_1 d\tau_2$$

$$\xi_1 = t_1\tau_1 \quad , \quad \xi_2 = t_2\tau_2$$
$$d\xi_1 = \tau_1 dt_1 \quad , \quad d\xi_2 = \tau_2 dt_1$$

$$\therefore \mathcal{Y}(s_1 s_2) = \int_0^\infty\int_0^\infty x(\tau_1 \tau_2) \int_0^\infty\int_0^\infty h(\xi_1, \xi_2)$$
$$\times \left(\frac{\xi_1}{\tau_1}\right)^{s_1-1} \left(\frac{\xi_2}{\tau_2}\right)^{s_2-1} d\xi_1 d\xi_2\, \frac{d\tau_1 d\tau_2}{\tau_1\tau_2}$$

$$= \int_0^\infty\int_0^\infty x(\tau_1\tau_2)\, \tau_1^{-s_1} \tau_2^{-s_2}\, d\tau_1 d\tau_2\, H(s_1 s_2)$$

$$= \int_0^\infty\int_0^\infty x(\tau_1 \tau_2)\, \tau_1^{(-s_1+1)-1} \tau_2^{-(s_2+1)-1}\, d\tau_1 d\tau_2$$
$$\times H(s_1 s_2)$$

$$= \mathcal{X}(1-s_1, 1-s_2)\, H(s_1, s_2)$$

c. $$y(t_1,t_2)=\int_0^\infty\int_0^\infty x(\tau_1,\tau_2)\,e^{-(t_1\tau_1+t_2\tau_2)}\,d\tau_1\,d\tau_2$$

$$h(t_1,t_2)= e^{-(t_1+t_2)}$$

$$H(s_1,s_2)=\int_0^\infty\int_0^\infty e^{-(t_1+t_2)}\,t_1^{s_1-1}\,t_2^{s_2-1}\,dt_1\,dt_2$$

$$=\int_0^\infty e^{-t_1}t_1^{s_1-1}\,dt_1\int_0^\infty e^{-t_2}t_2^{s_2-1}\,dt_2$$

$$=\Gamma(s_1)\,\Gamma(s_2)\;;\quad \mathrm{Re}\,s_1>0,\ \mathrm{Re}\,s_2>0$$

Note:

$$H(2,2)=\Gamma^2(2)\,\Gamma=(1!)^2=1$$

$$H(1,1)=\Gamma^2(1)=(0!)^2=1$$

$$H(3,3)=\Gamma^2(3)=(2!)^2=4$$

$$H(3,2)=\quad 2!=2$$

EE595

Name

Instructions:

1. This exam may be given to the student any time on Wed., Dec 10 (but not before).
2. The exam must reach Prof. Marks or Mr. Wu by Fri., 12-12-86, at 1 P.M. The receptionist in the EE main office can place it in a mail box - or the exam can be delivered personally. <u>No late exams will be accepted.</u>
3. The statement at the bottom of this page must be signed. Points will be shaved if any outside human help (other than Marks or Wu) is used. If such outside help is used, but not listed, procedures for academic misconduct discipline will be initiated.
4. Each problem is worth 25 points. When the tests are graded they can be picked up from the main office as usual. You can ask for your grade when you pick up your test.
5. Please submit your test with this as the cover page. Please staple.

The sources I have used for this test are listed on the back of this page.

X

date

1. A 2-D unilateral Mellin transform can be defined as:

$$\mathbb{X}(s_1, s_2) = \mathcal{M}\, x(t_1, t_2)$$
$$= \int_0^\infty \int_0^\infty x(t_1, t_2)\, t_1^{s_1-1}\, t_2^{s_2-1}\, dt_1 dt_2$$

(a) The Fourier transform is invariant to shift:

$$|\mathcal{F}\, x(t_1, t_2)| = |\mathcal{F}\, x(t_1-a, t_2-b)|$$

The Mellin transform, when evaluated at $s_1 = j\omega_1$ and $s_2 = j\omega_2$, is invariant to scale:

$$\left|\mathcal{M}\; x(t_1, t_2)\right|_{\substack{s_1=j\omega_1 \\ s_2=j\omega_2}} = \left|\mathcal{M}\; x\left(\frac{t_1}{A}, \frac{t_2}{B}\right)\right|_{\substack{s_1=j\omega_1 \\ s_2=j\omega_2}}$$

Prove this important result in pattern recognition.

(b) A "Mellin convolution" can be written as:

$$y(t_1, t_2) = \int_0^\infty \int_0^\infty x(\tau_1, \tau_2)\, h(t_1\tau_1, t_2\tau_2)\, d\tau_1 d\tau_2$$

As conventional convolutions are simplified by Fourier transformations, Mellin convolutions are simplified by Mellin transforms. Show how.

(c) The unilateral Laplace transform:

$$y(t_1, t_2) = \int_0^\infty \int_0^\infty x(\tau_1, \tau_2)\, e^{-(t_1\tau_1 + t_2\tau_2)}\, dt_1 dt_2$$

is a Mellin convolution. What is the "Mellin transfer function", $H(s_1, s_2)$, of this operation? Your answer should contain no integrals.

Hint: If you have the correct answer, then

$$H(2,2) = H(1,1) \text{ and } H^2(3,2) = H(3,3)$$

2. Choose a circularly sym. frequency response (other than a low pass filter) and, using the McClellan transform, generate the corresponding 2-D filter.

(a) Plot $H(\omega_1, 0)$

(b) Draw a signal flow graph for your filter using $F(\omega_1, \omega_2)$ filters.

3. page 106, problem 2.3

4. page 157, problem 3.10

Solutions

1.1.(a) $\vec{N}_1 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ $\vec{N}_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

$$\underline{N} = \begin{bmatrix} 5 & 2 \\ 1 & 5 \end{bmatrix}$$

(b) $\underline{P} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ $\hat{\underline{N}} = \underline{N}\,\underline{P} = \begin{bmatrix} 5 & 2 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 5 & -3 \\ 1 & 4 \end{bmatrix}$$

$\hat{\vec{N}}_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$ works!

Try $\hat{\underline{P}} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$ $\hat{\hat{\underline{N}}} = \begin{bmatrix} 5 & 2 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} -7 & 7 \\ -6 & 6 \end{bmatrix}$$

Both $(7, 6)'$ and $(-7, -6)$ work. Note, though, that $\det \hat{\hat{\underline{N}}} = 0$. Why? Because $\det \hat{\underline{P}} = 0$.

(c) This statement is true only if periodicity matrix is minimal. Note

$$|\det \underline{N}| = 23$$

$$|\det \hat{\underline{N}}| = 23$$

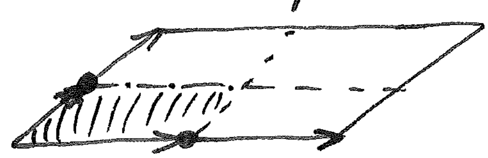
but $|\det \hat{\hat{\underline{N}}}| = 0$

If we had $\hat{\hat{\underline{P}}} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, then

$$\hat{\hat{\hat{\underline{N}}}} = \begin{bmatrix} 10 & 4 \\ 2 & 10 \end{bmatrix}$$

and $|\det \hat{\hat{\hat{\underline{N}}}}| = 92 = 4 \times 23$

but $\hat{\hat{\hat{\underline{N}}}}$ contains four minimal periods:



1.2.(a) $y[n_1 n_2] = x[n_1, n_2]\, x[n_1 - N, \; n_2] = T\, x[n_1 n_2]$

Linear? $\quad T\, a x[n_1 n_2] = a\, x[n_1 n_2]\, a x[n_1 - N, n_2]$

$a y[n_1 n_2] = a\, x[n_1 n_2]\, x[n_1 - N, n_2]$

not equal $\Rightarrow$ violates homogeneity $\Rightarrow$ <u>not</u> linear

Shift-invariant?

$T x[n_1 - k_1, n_2 - k_2] = x[n_1 - k_1, n_2 - k_2]\, x[n_1 - k_1 - N, n_2 - k_2]$

$y[n_1 - k_1, n_2 - k_2] = x[n_1 - k_1, n_2 - k_2]\, x[n_1 - k_1 - N, n_2 - k_2]$

they're equal $\Rightarrow$ shift-invariant

(b) $y[n_1 n_2] = \sum_{k_2 = -\infty}^{\infty} x[n_1 k_2] = L x_1 [n_1 n_2]$

Linear? $\quad L a x = a L x \Rightarrow$ homogeneity okay

$L x_1 + x_2 = L x_1 + L x_2 \Rightarrow$ additivity okay

$\Rightarrow$ Linear

Shift-invariant?

$L\, x_s[n_1 - k_1, n_2 - k_2] = \sum_{m_2 = -\infty}^{\infty} x[n_1 - k_1, m_2 - k_2]$

$= \sum_{\ell = -\infty}^{\infty} x[n_1 - k_1, \ell] \; ; \; \ell = m_2 - k_2$

$y[n_1 - k_1, n_2 - k_2] = \sum_{m_2} x[n_1 - k_1, m_2]$

They're equal $\Rightarrow$ shift-invariant

Note: $h[n_1, n_2] = \delta[n_1]$ (check it!)

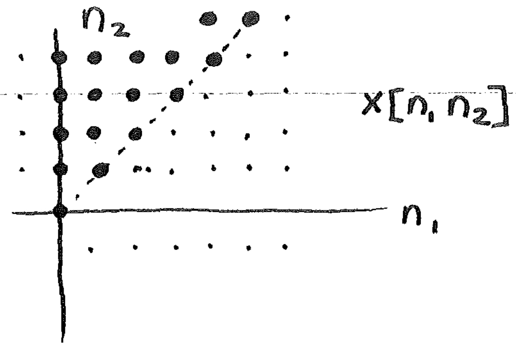
(c) $y[n_1 n_2] = \sum_{k_1 = -1}^{1} x[n_1 k_2] = T x[n_1 n_2]$

Linear $\rightarrow$ yes

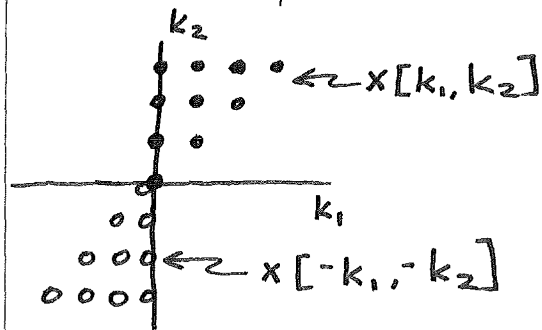
Shift invariant $\rightarrow$ no

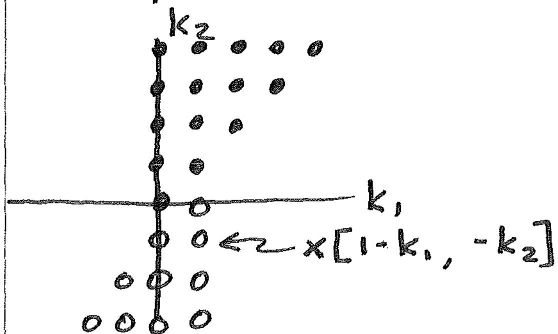
$h_{k_1 k_2}[n_1 n_2] = \delta[n_1 - k_1]\{\delta[k_2 - 1] + \delta[k_2] + \delta[k_2 + 1]\}$

1.3.



$$y = x ** x$$





For $n_1 = 0$

n_2	y
0	1
1	2
2	3
3	4
4	5
⋮	⋮

$y = (n_2 + 1)\,\mu[n_2]$

For $n_1 = 1$

n_2	y
0	0
1	2
2	4
3	6
4	8
5	10
⋮	⋮

$y = 2n_2\,\mu[n_2 - 1]$

A pattern!

For $n_1 = 2$, $y = 3(n_2 - 2)\,\mu[n_2 - 2]$

" $n_1 = 3$, $y = 4(n_2 - 3)\,\mu[n_2 - 3]$

⋮

$$\Rightarrow y[n_1, n_2] = (n_1 + 1)(n_2 - n_1)\,\mu[n_2 - n_1] \; ; \; n_1 \geq 0,\ n_2 \geq 0$$

$$= (n_1 + 1)(n_2 - n_1)\,\mu[n_2 - n_1]\,\mu[n_1]\,\mu[n_2]$$

Same region of support as x.

1.7.

(a) We know:

$$x[n_1,n_2] = \sum_{k_1}\sum_{k_2} x[k_1,k_2]\,\delta[n_1-k_1,\,n_2-k_2]$$

Note: $\delta[n-k] = \mu[n-k]-\mu[n-k-1]$

Thus:

$$x[n_1,n_2] = \sum_{k_1}\sum_{k_2} x[k_1,k_2]\,\{\mu[n_1-k_1]-\mu[n_1-k_1-1]\} \times \{\mu[n_2-k_2]-\mu[n_2-k_2-1]\}$$

$$= \sum_{k_1}\sum_{k_2} x[k_1,k_2]\,\mu[n_1-k_1,\,n_2-k_2]$$

$$- \sum_{p_1}\sum_{p_2} x[p_1,p_2]\,\mu[n_1-p_1,\,n_2-p_2-1]$$

$$- \sum_{q_1}\sum_{q_2} x[q_1,q_2]\,\mu[n_1-q_1-1,\,n_2-q_2]$$

$$+ \sum_{r_1}\sum_{r_2} x[r_1,r_2]\,\mu[n_1-r_1-1,\,n_2-r_2-1]$$

Set

$p_1=k_1,\; p_2+1=k_2$

$q_1+1=k_1,\; q_2=k_2$

$r_1+1=k_1,\; r_2+1=k_2$

Then:

$$x[n_1,n_2] = \sum_{k_1}\sum_{k_2}\{x[k_1,k_2]-x[k_1,k_2-1]-x[k_1-1,k_2] + x[k_1-1,k_2-1]\}\,\mu[n_1-k_1,\,n_2-k_2]$$

(b) $y[n_1,n_2] = L\,x[n_1,n_2]$

$$= L\sum_{k_1}\sum_{k_2}\{x[k_1,k_2]-x[k_1,k_2-1]-x[k_1-1,k_2]+x[k_1-1,k_2-1]\} \times \mu[n_1-k_1,\,n_2-k_2]$$

By additivity & homogeneity properties:

$$y[n_1,n_2] = \sum_{k_1}\sum_{k_2}\{x[k_1,k_2]-x[k_1,k_2-1]-x[k_1-1,k_2]+x[k_1-1,k_2-1]\} \times s[n_1-k_1,\,n_2-k_2]$$

where $s[n_1,n_2] = L\,\mu[n_1,n_2]$

$=$ unit step response

1.7 (cont)

(c) Note:

$$y[n_1, n_2] = \{x[n_1,n_2] - x[n_1, n_2-1] - x[n_1-1, n_2] + x[n_1-1, n_2-1]\} ** s[n_1,n_2]$$

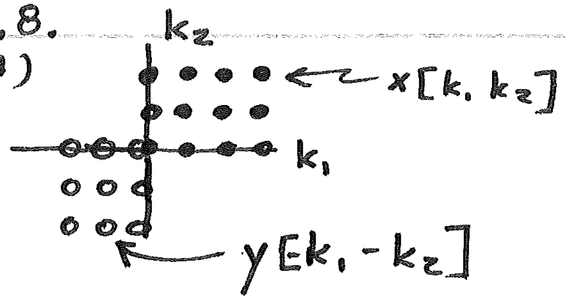
$$\stackrel{?}{=} \{s[n_1,n_2] - s[n_1, n_2-1] - s[n_1-1, n_2] + s[n_1-1, n_2-1]\} ** x[n_1, n_2]$$

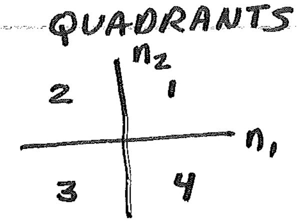
note:

$$w[n_1-a, n_2-b] ** v[n_1, n_2] = w[n_1, n_2] ** v[n_1-a, n_2-b]$$

Thus, using the distributive law of convolution (see prob 1-4 c), the two expressions <u>are</u> equal.

1.8.
(a)

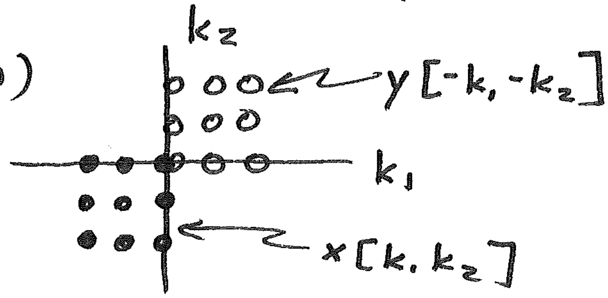




overlap <u>only</u> when $n_1 \geq 0$ and $n_2 \geq 0$

$\therefore$ support is only in first quadrant

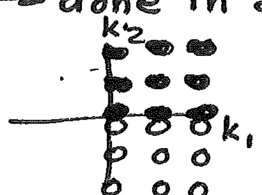
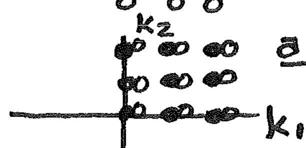
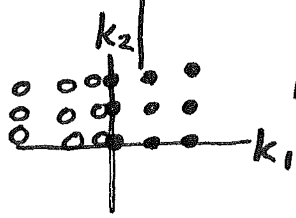
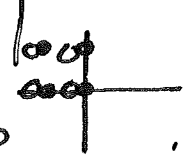
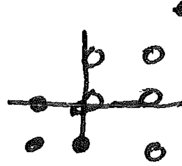
(b)



Overlap only when $n_1 \leq 0$ and $n_2 \leq 0$

$\Rightarrow$ support is in third quadrant

(c) Let's check them one @ a time

<u>Quadrants</u>			<u>Support Quadrants</u>
1 1	$\longrightarrow$ done in a		1
1 2		$n_2 \geq 0$	1+2
1 3		<u>all</u>	1+2+3+4
1 4		$n_1 \geq 0$	1+4
2 2	$\longrightarrow$ done in (b)		2
2 3		$n_1 \leq 0$	2+3
2 4		all	1+2+3+4
3 3		$n_1 \leq 0, n_2 \leq 0$	3
3 4		$n_2 \leq 0$	3+4
4 4	$\Longrightarrow$		4

Generalization:
1. Quadrants the same $\Longrightarrow$ Result in that Quadrant
2. Quadrants have only origin as a common point $\Longrightarrow$ Entire plane
3. Quadrants have a half-axis boundry $\Longrightarrow$ Support is union of the quadrants

In M.D

1.12 $$H(\vec{\omega}) = \sum_{n_1} \cdots \sum_{n_M} h[\vec{n}] e^{-j \vec{\omega}^T \vec{n}}$$

Let $\vec{k}$ = Vector of integers

$$H(\vec{\omega} + \vec{k}\, 2\pi) = \sum_{\vec{n}} h[\vec{n}]\, e^{-j(\vec{\omega} + 2\pi \vec{k})^T \vec{n}}$$

$$= \sum_{\vec{n}} h[\vec{n}] e^{-j\vec{\omega}^T \vec{n}} e^{j 2\pi \vec{k}^T \vec{n}}$$

But $\vec{k}^T \vec{n}$ = Integer $\Rightarrow$ $e^{j 2\pi \vec{k}^T \vec{n}} = 1$ and

$H(\vec{\omega} + \vec{k}\, 2\pi) = H(\vec{\omega})$

1.14 In m dimensions

Bracewell shows that:

$$\int_{\vec{x}} f(r)\, e^{\pm j2\pi \vec{u}^T \vec{x}}\, d\vec{x}$$

$$= \frac{2\pi}{q^{\frac{m}{2}-1}} \int_0^\infty f(r)\, J_{\frac{m}{2}-1}(2\pi q r)\, r^{\frac{m}{2}} dr$$

where $r = \|\vec{x}\| = \sqrt{\sum_{p=1}^{m} x_p^2}$ and $q = \|u\|$

Thus, for m dimensions [with $q = \|\vec{n}\|$]:

$$h[q] = \frac{2\pi}{q^{\frac{m}{2}-1}} \int_0^W r^{\frac{m}{2}} J_{\frac{m}{2}-1}(2\pi q r)\, dr \qquad (1)$$

From table of integrals:

$$\int \xi^{p+1} J_p(\xi)\, d\xi = \xi^{p+1} J_{p+1}(\xi)$$

In (1) set $\xi = 2\pi q r$

$$h[q] = \frac{2\pi}{q^{\frac{m}{2}-1}} \int_0^{2\pi q W} \left(\frac{\xi}{2\pi q}\right)^{\frac{m}{2}} J_{\frac{m}{2}-1}(\xi) \frac{d\xi}{2\pi q}$$

$$= \frac{(2\pi)^{-\frac{m}{2}}}{q^m} \int_0^{2\pi q W} \xi^{\frac{m}{2}} J_{\frac{m}{2}-1}(\xi)\, d\xi \;; \; p = \frac{m}{2}-1$$

$$= \frac{1}{(\sqrt{2\pi}\, q)^m}\, \xi^{\frac{m}{2}} J_{\frac{m}{2}}(\xi) \Big|_0^{2\pi q W}$$

$$= (\pi q W)^{\frac{m}{2}} J_{\frac{m}{2}}(2\pi q W) (\sqrt{2\pi}\, q)^{-m}$$

$$= \left(\frac{W}{2q}\right)^{\frac{m}{2}} J_{\frac{m}{2}}(2\pi W q)$$

$$; \; q = \sqrt{n_1^2 + \ldots + n_m^2}$$

1.15.(b) $x[n_1,n_2] = a^{n_1} b^{n_2} \delta[4n_1 - n_2]\, u[n_1]$

$$X(\omega_1,\omega_2) = \sum_{n_1=0}^{\infty} \sum_{n_2=-\infty}^{\infty} a^{n_1} b^{n_2} \delta[n_2 - 4n_1] e^{j\omega_1 n_1} e^{j\omega_2 n_2}$$

$$= \sum_{n_1=0}^{\infty} a^{n_1} b^{4n_1} e^{j\omega_1 n_1} e^{j\omega_2 4n_1}$$

$$= \sum_{n_1=0}^{\infty} [a b^4 e^{j\omega_1} e^{j4\omega_2}]^{n}$$

$$= \frac{1}{1 - a b^4 e^{j\omega_1} e^{j4\omega_2}} \quad \text{etc.}$$

(c) $$X(\omega_1\omega_2\omega_3) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=0}^{N-1} \sum_{n_3=0}^{M-1} u[n_1] a^{n_1} \times e^{j(\omega_1 n_1 + \omega_2 n_2 + \omega_3 n_3)}$$

$$= \sum_{n_1=0}^{\infty} (a e^{j\omega})^{n_1} \sum_{n_2=0}^{N-1} e^{j\omega_2 n_2} \sum_{n_3=0}^{M-1} e^{j\omega_3 n_3}$$

$$S = \sum_{n=0}^{P} a^n = 1 + a + \ldots + a^P$$

$$Sa = a + \ldots + a^{P+1}$$

$$(1-a)S = 1 - a^{P+1} \Rightarrow S = \frac{1-a^{P+1}}{1-a}$$

Thus $$X(\omega_1\omega_2\omega_3) = \frac{1}{1 - a e^{j\omega_1}} \; \frac{1 - e^{jN\omega_2}}{1 - e^{j\omega_2}} \; \frac{1 - e^{jM\omega_3}}{1 - e^{j\omega_3}}$$

$$= \frac{1 - a e^{-j\omega_1}}{1 + a^2 - 2a\cos\omega_1} \; \frac{e^{j\frac{N\omega_2}{2}} \sin\frac{N\omega_2}{2}}{e^{j\frac{\omega_2}{2}} \sin\frac{\omega_2}{2}} \; \frac{e^{j\frac{M\omega_3}{2}} \sin\frac{M\omega_3}{2}}{e^{j\frac{\omega_3}{2}} \sin\frac{\omega_3}{2}}$$

etc.

1.16. $y[n_1, n_2] = x[an_1 + bn_2,\ cn_1 + dn_2]$

$$Y(\omega_1, \omega_2) = \sum_{n_1}\sum_{n_2} y[n_1, n_2]\, e^{-j(\omega_1 n_1 + \omega_2 n_2)}$$

$$= \sum_{n_1}\sum_{n_2} x[an_1 + bn_2, cn_1 + dn_2]\, e^{-j(\omega_1 n_1 + \omega_2 n_2)}$$

$$= \sum_{\vec{n}} x[\underline{A}\,\vec{n}]\, e^{-j\vec{\omega}^T \vec{n}} = Y(\vec{\omega})$$

$$\underline{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Set $\vec{m} = \underline{A}\,\vec{n}$

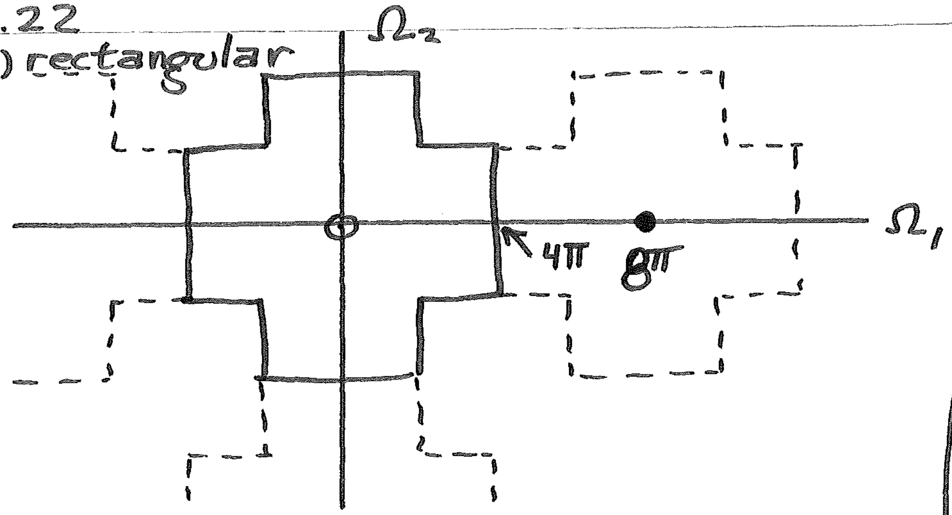
$$Y(\vec{\omega}) = \sum_{\vec{m}} x[\vec{m}]\, e^{-j\vec{\omega}^T \underline{A}^{-1}\vec{m}}$$

$$= \sum_{\vec{m}} x[\vec{m}]\, e^{-j(\underline{A}^{-1^T}\vec{\omega})^T \vec{m}}$$

$$= X(\underline{A}^{-1^T}\,\vec{\omega})$$

1.22

(a) rectangular



$$T_1 = \frac{2\pi}{2(4\pi)} = \frac{1}{4} = T_2$$

or

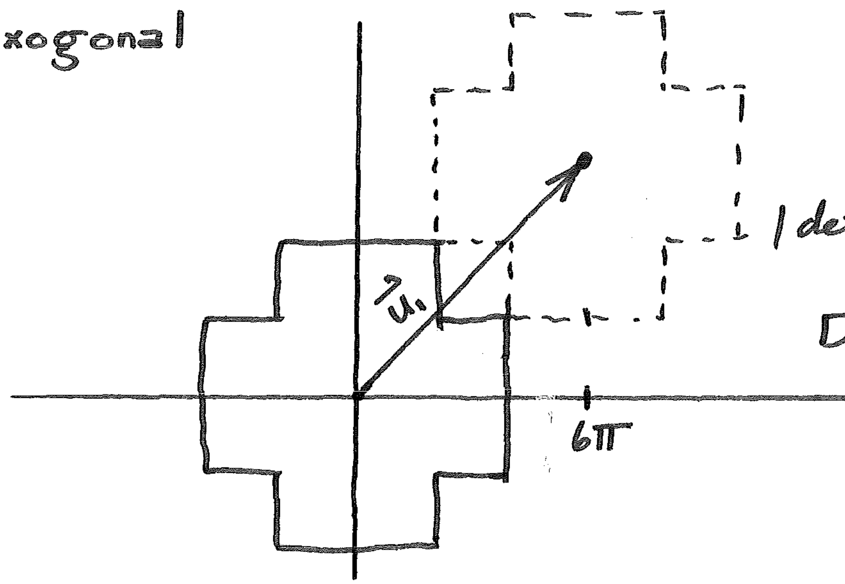
$$U = \begin{bmatrix} 8\pi & 0 \\ 0 & 8\pi \end{bmatrix}$$

$$\det U = (8\pi)^2$$

$$D = \frac{(8\pi)^2}{(2\pi)^2} = 16$$

$$D = \text{sampling density} = \frac{1}{T_1 T_2} = 16 \text{ samples}/m^2$$

(b) Hexogonal



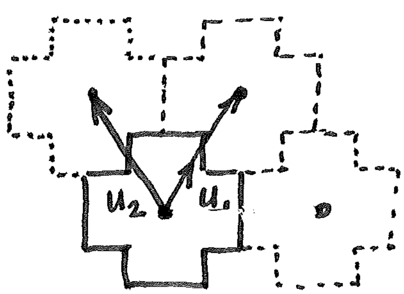
$$U = \begin{bmatrix} 6\pi & 6\pi \\ +6\pi & -6\pi \end{bmatrix}$$

$$|\det U| = 2(6\pi)^2$$

$$D = \frac{2(6\pi)^2}{(2\pi)^2} = 18 \frac{\text{samples}}{m^2}$$

worse

(C) Best:



$u_1 = (4\pi, 6\pi)$

$u_2 = (-4\pi, 6\pi)$

$$\underline{u} = \begin{bmatrix} 4\pi & -4\pi \\ 6\pi & 6\pi \end{bmatrix}$$

$$\det|\underline{u}| = 2(24\pi^2)$$

$$D = \frac{2 \cdot 24\pi^2}{4\pi^2} = 12$$

V

1.22 (cont).

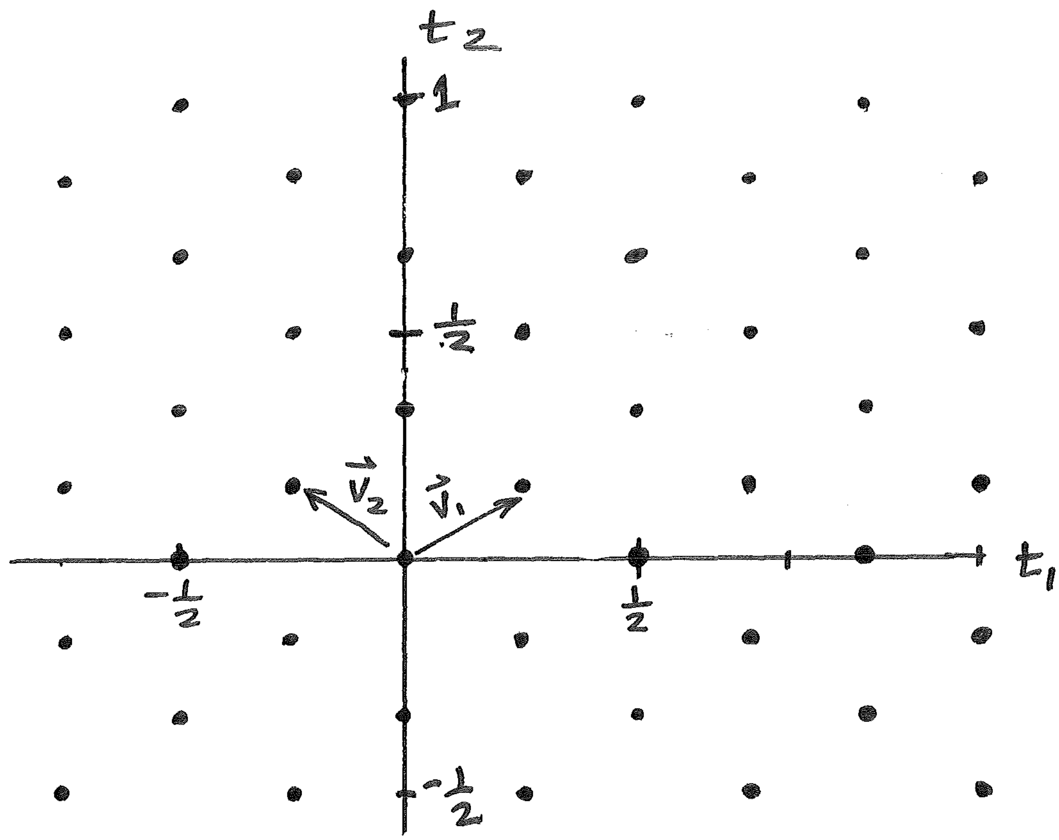
$$\vec{V} = 2\pi(\underline{U}^T)^{-1}$$

$$\underline{U}^T = \begin{bmatrix} 4\pi & 6\pi \\ -4\pi & 6\pi \end{bmatrix} \; ; \; (\underline{U}^T)^{-1} = \begin{bmatrix} \frac{6}{48\pi} & \frac{-6}{48\pi} \\ \frac{4}{48\pi} & \frac{4}{48\pi} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{8\pi} & -\frac{1}{8\pi} \\ \frac{1}{12\pi} & \frac{1}{12\pi} \end{bmatrix}$$

Thus: $\underline{V} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{6} & \frac{1}{6} \end{bmatrix}$

$$= \begin{bmatrix} \frac{3}{12} & -\frac{3}{12} \\ \frac{2}{12} & \frac{2}{12} \end{bmatrix} \Rightarrow \vec{V}_1 = \begin{bmatrix} \frac{3}{12} \\ \frac{2}{12} \end{bmatrix} \quad \vec{V}_2 = \begin{bmatrix} -\frac{3}{12} \\ \frac{2}{12} \end{bmatrix}$$



2-1 In M-D

$$\tilde{x}[\vec{n}] = \frac{1}{|\det \underline{N}|} \sum_{\vec{k} \in R_N} \tilde{X}[\vec{k}] e^{j2\pi \vec{n}^T \underline{N}^{-1} \vec{k}}$$

$$= \frac{1}{|\det \underline{N}|} \sum_{\vec{k} \in R_N} \sum_{\vec{m} \in R_N} \tilde{x}[\vec{m}] e^{-j2\pi \vec{m}^T \underline{N}^{-1} \vec{k}} e^{j2\pi \vec{n}^T \underline{N}^{-1} \vec{k}}$$

$$= \frac{1}{|\det \underline{N}|} \sum_{\vec{m} \in R_N} \tilde{x}[\vec{m}] \sum_{\vec{k} \in R_N} e^{j2\pi (\vec{n}-\vec{m})^T \underline{N}^{-1} \vec{k}} \qquad (1)$$

But

$$\sum_{\vec{k} \in R_N} e^{j2\pi(\vec{n}-\vec{m})^T \underline{N}^{-1} \vec{k}} = \sum_{k_1=0}^{N_1-1} e^{j2\pi(n_1-m_1)k_1/N_1} \cdots \sum_{k_M=0}^{N_M-1} e^{j2\pi(n_M-m_M)k_M/N_M}$$

$$= N_1 N_2 \ldots N_M \ \delta(\vec{n}-\vec{m})$$

$$= \det \underline{N} \ \delta(\vec{n}-\vec{m})$$

Substitute into (1), sift, and Q.E.D.

2.2. In M dimensions

(a) $$\tilde{x}[\vec{n}-\vec{m}] \longleftrightarrow \tilde{X}[\vec{k}]\, e^{-j2\pi \vec{m}^T \underline{N}^{-1}\vec{k}}$$

(b) If $\vec{p} = [p_1\ p_2 \cdots p_M]^T$

Define $\vec{p}_r = [p_M\ p_{M-1} \cdots p_1]^T$

Then $\tilde{x}[\vec{n}_r] \longleftrightarrow \tilde{X}[\vec{k}_r]$

(c) $$\tilde{Y}(\vec{k}) = \sum_{\vec{n}} \tilde{x}^*[\vec{n}]\, e^{-j2\pi \vec{n}^T \underline{N}^{-1}\vec{k}}$$

$$= \left[\sum_{\vec{n}} \tilde{x}[\vec{n}]\, e^{+j2\pi \vec{n}^T \underline{N}^{-1}\vec{k}}\right]^*$$

$$= \left[\sum_{\vec{n}} \tilde{x}[\vec{n}]\, e^{-j2\pi \vec{n}^T \underline{N}^{-1}(-\vec{k})}\right]^*$$

$$= \tilde{X}^*[-\vec{k}]$$

(d) $$\tilde{Y}[\vec{k}] = \sum_{\vec{m}} \tilde{x}[-\vec{m}]\, e^{-j2\pi \vec{m}^T \underline{N}^{-1}\vec{k}}$$

Variable substitution: $\vec{n} = -\vec{m}$

$$\tilde{Y}[\vec{k}] = \sum_{\vec{n}} \tilde{x}[\vec{n}]\, e^{-j2\pi \vec{n}^T \underline{N}^{-1}(-\vec{k})}$$

$$= \tilde{X}[-\vec{k}]$$

2-4 $\quad N_2 = 5 = N_1$

(a) $$\mathbb{X}(k_1, k_2) = \underbrace{\sum_{n_1}\sum_{n_2}}_{\text{period}} x[n_1, n_2]\, e^{-j2\pi\left(\frac{n_1k_1}{5} + \frac{n_2k_2}{5}\right)}$$

$$= 1 + e^{-j2\pi\frac{k_2}{5}} + e^{j2\pi\frac{k_2}{5}}$$

$$+ e^{-j2\pi k_1/5} + e^{j2\pi k_1/5}$$

$$= 1 + 2\cos\frac{2\pi k_2}{5} + 2\cos\frac{2\pi k_1}{5}$$

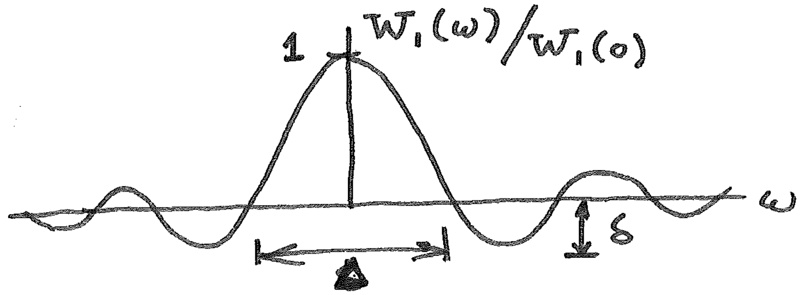
(b) $$\mathbb{X}[k_1, k_2] = e^{-j2\pi\frac{k_1+k_2}{5}} + e^{j2\pi\frac{k_1+k_2}{5}}$$

$$+ e^{-j2\pi\frac{k_1-k_2}{5}} + e^{j2\pi\frac{k_1-k_2}{5}}$$

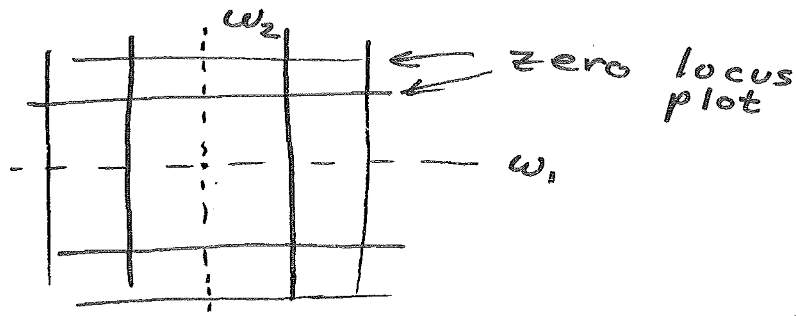
$$= 2\cos 2\pi\frac{k_1+k_2}{5} + 2\cos 2\pi\frac{k_1-k_2}{5}$$

$$= 4\cos\frac{2\pi k_1}{5}\cos\frac{2\pi k_2}{5}$$

3-5. Assume $W_1(\omega)$ has zero crossings:



$$W_2(\omega_1, \omega_2) = W_1(\omega_1) W_1(\omega_2)$$



$\Delta_2 = \Delta$ in ω_1 & ω_2 directions

$(\Delta_2)_{max} = \sqrt{2}\,\Delta$ (@ 45°)

$\delta_2 = \delta$ in ω_1 & ω_2 directions

@ 45°, roughly,

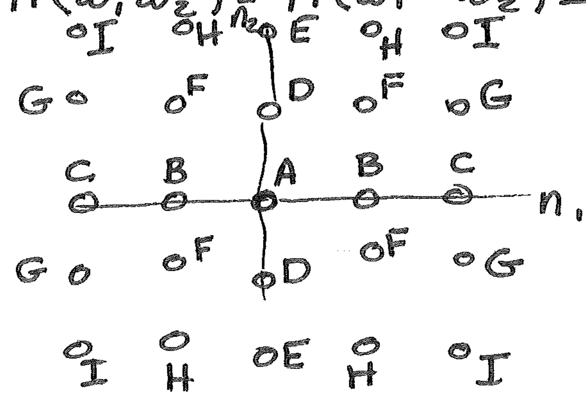
$$\delta_2 = \delta_1^2$$

3.8(a) $H(\vec{\omega})$ real $\Rightarrow$ h is even & real

$$h[n_1 n_2] = h[-n_1, -n_2]$$

$$H(\omega_1 \omega_2) = H(-\omega_1, \omega_2) \Rightarrow h[n_1 n_2] = h[n_1, +n_2]$$

$$H(\omega_1 \omega_2) = H(\omega_1 -\omega_2) \Rightarrow h[n_1 n_2] = h[n_1 \; -n_2]$$



$\Leftarrow$ 5 × 5

$\Rightarrow$ 9 Degrees of freedom

(b) $$H(\omega_1 \omega_2) = \sum_{n_2=-2}^{2} \sum_{n_1=-2}^{2} h[n_1 n_2]\, e^{-j(\omega_1 n_1 + \omega_2 n_2)}$$

$$\begin{aligned}
h[n_1 n_2] = {} & A\delta[n_1 \cdot n_2] + B[\delta(n_1-1) + \delta(n_1+1)]\delta[n_2] \\
& + C[\delta(n_1-2) + \delta[n_1+2]]\delta[n_2] \\
& + D[\delta(n_2-1) + \delta(n_2+1)]\delta[n_1] \\
& + E[\delta(n_2-2) + \delta(n_2+2)]\delta[n_1] \\
& + F[\delta(n_1-1, n_2-1) + \delta(n_1+1, n_2-1) \\
& \quad + \delta(n_1+1, n_2+1) + \delta[n_1-1, n_2+1]) \\
& + G[\delta[n_1-2, n_2-1] + \delta[n_1+2, n_2-1] \\
& \quad + \delta[n_1+2, n_2+1] + \delta[n_1-2, n_2+1]) \\
& + H[\delta[n_1-1, n_2-2] + \delta[n_1+1, n_2-2] \\
& \quad + \delta[n_1+1, n_2+2] + \delta[n_1-1, n_2+2]] \\
& + I[\delta[n_1-2, n_2-2] + \delta[n_1-2, n_2+2] \\
& \quad + \delta[n_1+2, n_2-2] + \delta[n_1+2, n_2+2])
\end{aligned}$$

3-8 (cont)

Thus:

$$H(\omega_1,\omega_2) = A + 2B\cos\omega_1 + 2C\cos 2\omega_1 + 2D\cos\omega_2 + 2E\cos 2\omega_2$$
$$+ 2F\left[\cos(\omega_1+\omega_2) + \cos(\omega_1-\omega_2)\right]$$
$$+ 2G\left[\cos(2\omega_1+\omega_2) + \cos(2\omega_1-\omega_2)\right]$$
$$+ 2H\left[\cos(\omega_1+2\omega_2) + \cos(\omega_1-2\omega_2)\right]$$
$$+ 2I\left[\cos(2\omega_1+2\omega_2) + \cos(2\omega_1+2\omega_2)\right]$$

$$= A + 2B\cos\omega_1 + 2C\cos 2\omega_1 + 2D\cos\omega_2 + 2E\cos 2\omega_2$$
$$+ 4F\cos\omega_1\cos\omega_2 + 4G\cos 2\omega_1\cos\omega_2$$
$$+ 4H\cos\omega_1\cos 2\omega_2 + 4I\cos 2\omega_1\cos 2\omega_2$$

$$= \sum_{p=1}^{9} a[p]\,\phi_p(\omega_1,\omega_2)$$

$$\phi_p(\omega_1,\omega_2) = \cos(n\omega_1)\cos(m\omega_1) \qquad 0 \le n,m \le 2$$

or

$$H(\omega_1,\omega_2) = \sum_{n=0}^{2}\sum_{m=0}^{2} a_{nm}\cos n\omega_1 \cos m\omega_2$$

(3-8) cont

(c) Given $i[n_1 n_2]$

$$E_2 = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\!\!\int \left| H(\omega_1 \omega_2) - I(\omega_1 \omega_2)\right|^2 d\omega_1 d\omega_2$$

$$= \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \left| \sum_{n_1=0}^{2}\sum_{n_2=0}^{2} a_{nm} \cos n\omega_1 \cos n\omega_2 - I(\omega_1 \omega_2)\right|^2 d\omega_1 d\omega_2$$

Thus:

$$\sum_{n_1=0}^{2}\sum_{n_2=0}^{2} a_{nm}\, \phi_{nm,k\ell} = I_{k\ell}$$

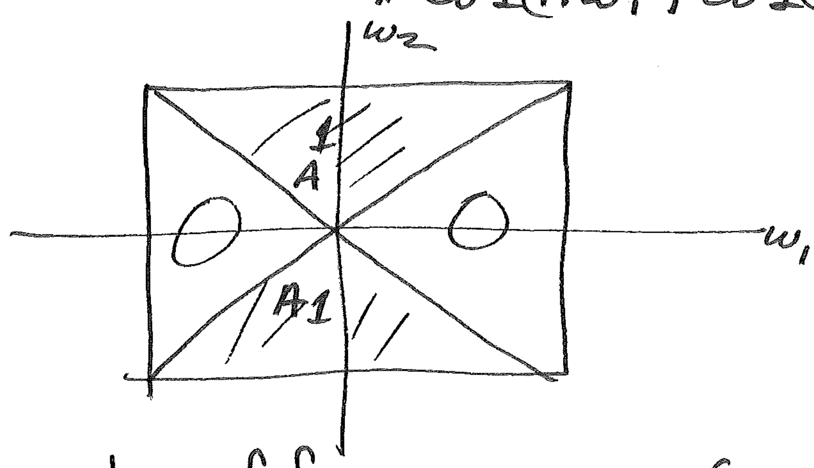
$$\phi_{nm,k\ell} = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \cos(n\omega_1)\cos(m\omega_2) \times \cos(k\omega_1)\cos(\ell\omega_2)$$

$$= \delta[n-k,\ m-\ell]$$

Thus:

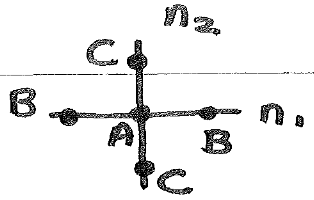
$$a_{nm} = I_{nm}$$

$$= \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} I(\omega_1 \omega_2) \times \cos(n\omega_1)\cos(m\omega_2)\, d\omega_1 d\omega_2$$



$$a_{nm} = \frac{1}{(2\pi)^2}\iint_A \cos(n\omega_1)\cos(m\omega_2)\, d\omega_1 d\omega_2$$

3.10.



$$H(\omega_1,\omega_2) = A + 2B\cos\omega_1 + 2C\cos\omega_2$$

$$E = \iint_{-\pi}^{\pi} \left[I(\vec{\omega}) - H(\vec{\omega})\right]^2 d\vec{\omega}$$

$$= \int_{-\pi}^{\pi}\!\!\int \left[I(\vec{\omega}) - A - 2B\cos\omega_1 - 2C\cos\omega_2\right]^2 d\vec{\omega}$$

$$\frac{\delta E}{\delta A} = \iint_{-\pi}^{\pi} 2\left[I(\vec{\omega}) - A - 2B\cos\omega_1 - 2C\cos\omega_2\right](-1)\,d\vec{\omega}$$

$$\Rightarrow 2\left[4ab - A(2\pi)^2\right] = 0 \Rightarrow A = \frac{4ab}{(2\pi)^2} = \frac{ab}{\pi^2}$$

$$\frac{\delta E}{\delta B} = \iint_{-\pi}^{\pi} 2\left[I(\vec{\omega}) - A - 2B\cos\omega_1 - 2C\cos\omega_2\right](2\cos\omega_1)\,d\vec{\omega}$$

$$= 0 \Rightarrow \iint_{-\pi}^{\pi} I(\vec{\omega})\cos\omega_1\, d\omega_1 d\omega_2$$

$$= \iint_{-\pi}^{\pi} \left[A\cancel{\cos\omega_1} + 2B\cos^2\omega_1 + 2C\cancel{\cos\omega_1}\cos\omega_2\right] d\vec{\omega}$$

or

$$2b\int_{-a}^{a} \cos\omega_1\, d\omega_1 = B \iint_{-\pi}^{\pi} (1 + \cos 2\omega_1)\, d\omega_1$$

$$2b \sin\omega_1 \Big|_{-a}^{a} = (2\pi)^2 B$$

$$\cancel{4}b\sin a = (\cancel{2}\pi)^2 B \quad \Longrightarrow B = \frac{b\sin a}{\pi^2}$$

$$C = \frac{a\sin b}{\pi^2}$$

3.17.

$$H(\vec{\omega}) = \sum_{n=0}^{N} a[n]\, T_n[F(\vec{\omega})]$$

$$= \sum_{n=0}^{N} a[n] \sum_{m=0}^{n} b_{mn}\, F^m(\vec{\omega})$$

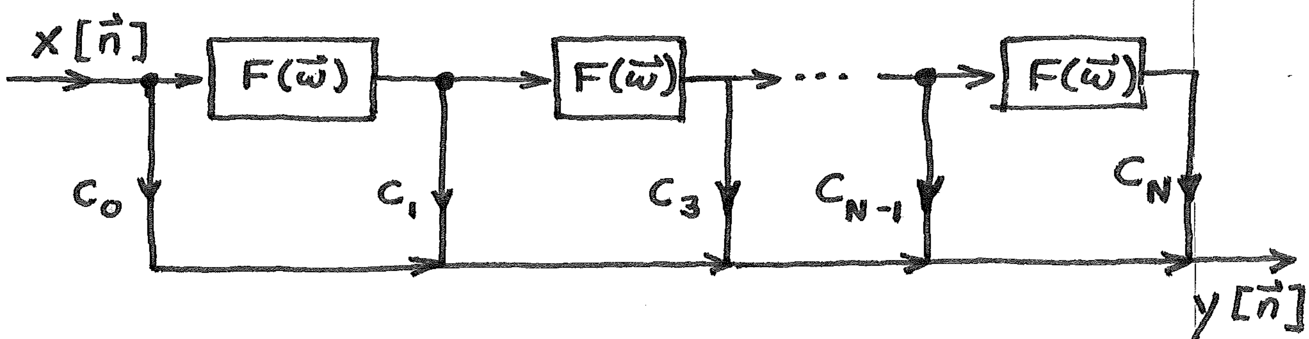
coefficients
of Chebychev Polynomials

$$= \sum_{n=0}^{N} a[n] \sum_{m=0}^{N} b_{mn}\, \mu[n-m]\, F^m(\vec{\omega})$$

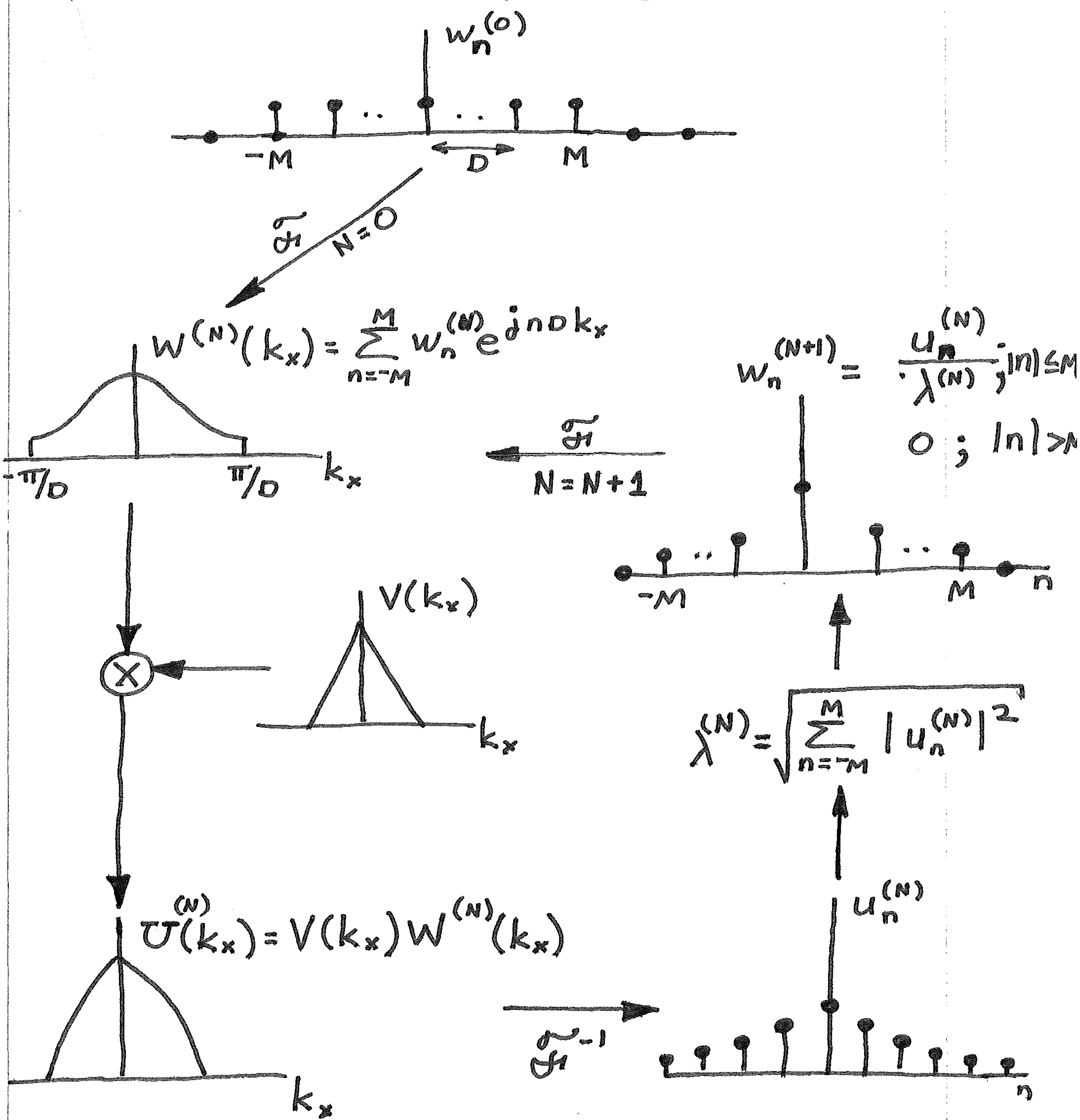
$$= \sum_{m=0}^{N} F^m(\vec{\omega}) \sum_{n=0}^{N} a[n]\, \mu[n-m]\, b_{mn}$$

$$= \sum_{m=0}^{N} F^m(\vec{\omega}) \sum_{n=m}^{N} a[n]\, b_{mn}$$

$$= \sum_{m=0}^{N} c_m F^m(\vec{\omega}) \; ; \quad c_m = \sum_{n=m}^{N} a[n]\, b_{mn}$$



Iterative Linear Array Design



Multiplying periodic functions $\Rightarrow$ circularly convolve their Fourier coefficients. Thus:

$$u_n^{(N)} = \sum_{m=-M}^{M} w_m^{(N)} v_{n-m} \tag{1}$$

Note: equivalent to a matrix multipication. Corresponding eigen. solution:

$$\lambda_k \psi_n[k] = \sum_{m=-M}^{M} \psi_m[k] v_{n-m} \quad ; \quad |k| \leq M \tag{2}$$

Then:

$$w_n^{(0)} = \sum_{k=0}^{2M} a_k \psi_n[k] \tag{3}$$

where

$$a_k = \sum_{n=-M}^{M} w_n^{(0)} \psi_n[k] \tag{4}$$

Note that we can write (1) & (2) in matrix form:

$$\overrightarrow{u^{(N)}} = \underline{V} \, \overrightarrow{w^{(N)}} \tag{5}$$

$$\lambda_k \overrightarrow{\psi[k]} = \underline{V} \, \overrightarrow{\psi[k]} \tag{6}$$

Going to the algorithm:

$$\begin{aligned} w_n^{(1)} &= \frac{1}{\lambda^{(0)}} \underline{V} \, w_n^{(0)} \\ &= \frac{1}{\lambda^{(0)}} \underline{V} \sum_{k=0}^{2M} a_k \psi_n[k] \\ &= \frac{1}{\lambda^{(0)}} \sum_{k=0}^{2M} a_k \lambda_k \psi_n[k] \end{aligned} \tag{7}$$

and:

$$w_n^{(2)} = \frac{1}{\lambda^{(1)}} \sum_{k=0}^{2M} a_k \lambda_k^2 \psi_n[k] \tag{8}$$

or, in general:

$$w_n^{(N+1)} = \frac{1}{\lambda^{(N)}} \sum_{k=0}^{2M} a_k \lambda_k^{N+1} \psi_n[k] \tag{9}$$

$$= \frac{\sum_{k=0}^{2M} a_k \lambda_k^{N+1} \psi_n[k]}{\left[\sum_{n=-M}^{M} \left|\sum_{k=0}^{2M} a_k \lambda_k^{N+1} \psi_n[k]\right|^2\right]^{1/2}} \tag{10}$$

$\underline{V}$ is Hermetian. We can order:

$$\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_{2M}$$

As $N \to \infty$, the λ_{2M} term is (10) will dominate all others. Thus:

$$w_n^{(N)} \xrightarrow[N\to\infty]{} \frac{a_{2M} \lambda_{2M}^{N+1} \psi_n[2M]}{\left[\sum_{m=-M}^{M} \left(a_{2M} \lambda_{2M}^{N+1} \psi_m[2M]\right)^2\right]^{1/2}} \tag{11}$$

$$= \frac{\psi_n[2M]}{\|\psi_n[2M]\|_M} \tag{12}$$

Note also that

$$\lambda^{(N)} \xrightarrow[N\to\infty]{} \lambda_{2M} \tag{13}$$

Our iterative result maximizes

$$\alpha = \int_{-\pi/D}^{\pi/D} V(k_x)\,|W(k_x)|^2 dk_x \qquad (14)$$

when $\int_{-\pi/D}^{\pi/D} |W(k_x)|^2 dk_x = 1$ (15)

Proof: Let

$$w_n = \sum_{k=0}^{2M} b_k \psi_n[k] \qquad (16)$$

Then:

$$W(k_x) = \sum_{n=-M}^{M} \sum_{k=0}^{2M} b_k \psi_n[k] e^{jnDk_x} \qquad (17)$$

and

$$\int_{-\pi/D}^{\pi/D} V(k_x)\,|W(k_x)|^2 dk_x = \sum_{n=-M}^{M} \sum_{m=-M}^{M} \sum_{k=0}^{2M} \sum_{\ell=0}^{2M} b_k b_\ell^* \psi_n[k] \psi_m^*[\ell] \int_{-\pi/D}^{\pi/D} V(k_x) e^{j(n-m)Dk_x} dk_x \qquad (18)$$

$$= \sum_{nmk\ell} b_k b_\ell^* \psi_n[k] \psi_m^*[\ell] V_{n-m} \qquad (19)$$

Perform the **n** sum (using (2)):

$$\alpha = \sum_{mk\ell} b_k b_\ell^* \psi_m^*[k] \psi_m[\ell] \lambda_k$$

$$= \sum_{\ell=0}^{2M} b_\ell^* \sum_{k=0}^{2M} b_k \lambda_k \underbrace{\sum_{m=-M}^{M} \psi_m[\ell] \psi_m^*[k]}_{\delta[k-\ell]}$$

$$= \sum_{k=0}^{2M} |b_k|^2 \lambda_k \qquad (20)$$

Note (15) is equivalent to

$$\sum_{k=0}^{2M} |b_k|^2 = 1$$

Thus, to maximize α, we choose

$$b_k = \begin{cases} 0 & ; 0 \le k < 2M \\ 1 & ; 2M \end{cases}$$

since λ_{2M} is the largest. From (16), our best array is then

$$w_n = \psi_n[2N]$$

The corresponding maximum α is λ_{2N}.

Note: For $V(k_x) = p_T(k_x)$, the ψ_n's are Digital Prolate Functions (See Papoulis, <u>SIGNAL ANALYSIS</u>, pp. 212-214)

Noise Sensitivity of the Multidimensional Sampling Theorem

The sampling theorem is:

$$x_a(\vec{t}) = \sum_{\vec{n}} x_a(\underline{v}\vec{n})\, f(\vec{t} - \underline{v}\vec{n}) \tag{1}$$

where:

$$f(\vec{t}) = \frac{|\det \underline{v}|}{(2\pi)^N} \int_{\mathcal{B}} e^{j\vec{\Omega}'\vec{t}}\, d\vec{\Omega} \tag{2}$$

and $\mathcal{B}$ is any region containing the zeroth order spectrum (see Fig 3 of paper reprint by Marks). Instead of $x_a(\underline{v}\vec{n})$, suppose we have the noisy samples:

$$x_a(\underline{v}\vec{n}) + \xi(\underline{v}\vec{n})$$

Substituting this into (1) in lieu of $x_a(\underline{v}\vec{n})$ gives a result of

$$x_a(\vec{t}) + \eta(\vec{t})$$

where:

$$\eta(\vec{t}) = \sum_{\vec{n}} \xi(\underline{v}\vec{n})\, f(\vec{t} - \underline{v}\vec{n})$$

Clearly, if $\xi(\vec{t})$ is zero mean, so is $\eta(\vec{t})$:

$$E\,\eta(\vec{t}) = \sum_{\vec{n}} E[\xi(\underline{v}\vec{n})]\, f(\vec{t} - \underline{v}\vec{n})$$
$$= 0$$

Thus,

$$\operatorname{var} \eta(\vec{t}) = \overline{\eta^2(\vec{t})}$$
$$= \sum_{\vec{n}} \sum_{\vec{m}} R_\xi[\underline{v}(\vec{n} - \vec{m})] \times f(\vec{t} - \underline{v}\vec{n})\, f(\vec{t} - \underline{v}\vec{m}) \tag{3}$$

where $\overline{\eta^2} = E\,\eta^2$ and the data's (stationary) autocorrelation is:

$$R_\xi(\vec{t} - \vec{\tau}) = E\, \xi(\vec{t})\, \xi(\vec{\tau})$$

For discrete white noise:

$$R_\xi(\underline{v}\vec{n}) = \overline{\xi^2}\ \delta(\vec{n})$$

Substituting into (3) gives:

$$\text{var}\ n(\vec{t}) = \overline{\xi^2} \sum_{\vec{n}} f^2(\vec{t} - \underline{v}\vec{n}) \tag{4}$$

Using (1) for $x_a(\vec{t}) = f(\vec{\tau} - \vec{t})$ (where $\vec{\tau}$ is a fixed number) gives

$$f(\vec{t} - \vec{\tau}) = \sum_{\vec{n}} f(\tau - \underline{v}\vec{n})\, f(\vec{t} - \underline{v}\vec{n})$$

Setting $\vec{t} = \vec{\tau}$ reduces (4) to

$$\text{var}\ n(\vec{t}) = \overline{\xi^2}\ f(\vec{0})$$

From (2):

$$f(\vec{0}) = \frac{|\det \underline{V}|}{(2\pi)^N} \int_{\mathcal{B}} d\vec{\Omega}$$

$$= B/C$$

where:

$$B = \frac{1}{(2\pi)^N} \int_{\mathcal{B}} d\vec{\Omega}$$

$= \text{Area of } \mathcal{B}$ (in $(hz)^N$)

and

$$C = \frac{1}{|\det \underline{V}|} = \frac{|\det \underline{U}|}{(2\pi)^N}$$

$= \text{Area of a cell } \mathcal{C}$ (in $(hz)^N$)

$\therefore$ to minimize noise, choose

$$\mathcal{B} = \mathcal{A} = \text{region of support}$$

Homework:

Chapt 1: 12 (in N dimensions)

16

14 (in N dimensions)

15

17

22

24

20 (in N dimensions)*

Extra Problem

*Hint: Bracewell shows that

$$\int_{\vec{x}} f(r) e^{\pm j 2\pi \vec{u}^T \vec{x}} d\vec{x}$$

$$= \frac{2\pi}{q^{\frac{m}{2}-1}} \int_0^\infty r^{\frac{n}{2}} f(r)\, J_{\frac{m}{2}-1}(2\pi q r) dr$$

where:

$$r = \sqrt{\sum_{n=1}^{N} x_n^2} = \|\vec{x}\|$$

and

$$q = \|\vec{u}\|$$

Extra Problem: Show that a function with no nonzero components outside a circle of radius $2\pi B$ on the Ω_1, Ω_2 plane obeys the following sampling theorem:

$$x(t_1, t_2) = \sum_{n_1=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x\left(\frac{n_1}{2B}, \frac{n_2}{2B}\right) \left\{ \frac{\pi}{2} \frac{J_1\left[2\pi B \sqrt{\left(t_1 - \frac{n_1}{2B}\right)^2 + \left(t_2 - \frac{n_2}{2B}\right)^2}\right]}{2\pi B \sqrt{\left(t_1 - \frac{n_1}{2B}\right)^2 + \left(t_2 - \frac{n_2}{2B}\right)^2}} \right\}$$

EE595 Test solutions

1.a. $\mathbb{X}(j\omega_1, j\omega_2) = \int_0^\infty \int_0^\infty x(t_1,t_2)\, t_1^{j\omega_1 - 1} t_2^{j\omega_2 - 1}\, dt_1 dt_2$

$\mathcal{M}\, x(\frac{t_1}{A}, \frac{t_2}{B}) = \int_0^\infty \int_0^\infty x(\frac{t_1}{A}, \frac{t_2}{B})\, t_1^{j\omega_1 - 1} t_2^{j\omega_2 - 1}\, dt_1 dt_2$

$$\tau_1 = t_1/A \quad , \quad \tau_2 = t_2/B$$

$$\mathcal{M}\, x(\tfrac{t_1}{A}, \tfrac{t_2}{B}) = \int_0^\infty\int_0^\infty x(\tau_1 \tau_2)\, (\tau_1 A)^{j\omega_1 - 1} (\tau_2 B)^{j\omega_2 - 1} \times A\, d\tau_1\, B\, d\tau_2$$

But $|A^{j\omega}| = |e^{j(\ln A)\omega}| = 1$. Thus:

$$|\mathcal{M}\; x(\tfrac{t_1}{A}, \tfrac{t_2}{B})| = \left| \int_0^\infty\int x(\tau_1 \tau_2)\, \tau_1^{j\omega_1 - 1} \tau_2^{j\omega_2 - 1}\, d\tau_1 d\tau_2 \right|$$

$$= |\mathcal{M}\, x(t_1, t_2)|$$

b. $$y(t_1, t_2) = \int_0^\infty\int x(\tau_1, \tau_2)\, h(t_1\tau_1, t_2\tau_2)\, d\tau_1 d\tau_2$$

$$\mathbb{Y}(s_1, s_2) = \int_0^\infty \int_0^\infty y(t_1 t_2)\, t_1^{s_1 - 1} t_2^{s_2 - 1}\, dt_1 dt_2$$

$$= \int_{\tau_1=0}^\infty \int_{\tau_2=0}^\infty x(\tau_1 \tau_2) \left[\int_{t_1=0}^\infty \int_{t_2=0}^\infty h(t_1\tau_1, t_2\tau_2)\, t_1^{s_1 - 1} t_2^{s_2 - 1}\, dt_1 dt_2 \right] d\tau_1 d\tau_2$$

$$\xi_1 = t_1\tau_1 \quad , \quad \xi_2 = t_2\tau_2$$

$$d\xi_1 = \tau_1 dt_1 \quad , \quad d\xi_2 = \tau_2 dt_1$$

$$\therefore \mathbb{Y}(s_1 s_2) = \int_0^\infty \int_0^\infty x(\tau_1 \tau_2) \int_0^\infty \int_0^\infty h(\xi_1, \xi_2) \times \left(\frac{\xi_1}{\tau_1}\right)^{s_1 - 1} \left(\frac{\xi_2}{\tau_2}\right)^{s_2 - 1} d\xi_1 d\xi_2\, \frac{d\tau_1 d\tau_2}{\tau_1 \tau_2}$$

$$= \int_0^\infty \int_0^\infty x(\tau_1 \tau_2)\, \tau_1^{-s_1} \tau_2^{-s_2}\, d\tau_1 d\tau_2\, H(s_1 s_2)$$

$$= \int_0^\infty \int_0^\infty x(\tau_1 \tau_2)\, \tau_1^{(-s_1+1)-1} \tau_2^{-(s_2+1)-1}\, d\tau_1 d\tau_2 \times H(s_1 s_2)$$

$$= \mathbb{X}(1 - s_1, 1 - s_2)\, H(s_1, s_2)$$

c. $$y(t_1, t_2) = \int_0^\infty \int_0^\infty x(\tau_1, \tau_2)\, e^{-(t_1\tau_1 + t_2\tau_2)}\, d\tau_1 d\tau_2$$

$$h(t_1, t_2) = e^{-(t_1 + t_2)}$$

$$H(s_1 s_2) = \int_0^\infty \int_0^\infty e^{-(t_1 + t_2)}\, t_1^{s_1 - 1}\, t_2^{s_2 - 1}\, dt_1 dt_2$$

$$= \int_0^\infty e^{-t_1} t_1^{s_1 - 1} dt_1 \int_0^\infty e^{-t_2} t_2^{s_2 - 1} dt_2$$

$$= \Gamma(s_1)\,\Gamma(s_2)\;; \quad Re\, s_1 > 0$$

Note: $$Re\, s_2 > 0$$

$$H(2,2) = \Gamma^2(2)\,\Gamma = (1!)^2 = 1$$

$$H(1,1) = \Gamma^2(1) = (0!)^2 = 1$$

$$H(3,3) = \Gamma^2(3) = (2!)^2 = 4$$

$$H(3,2) = 2! = 2$$

name

EE 595 Midterm #1

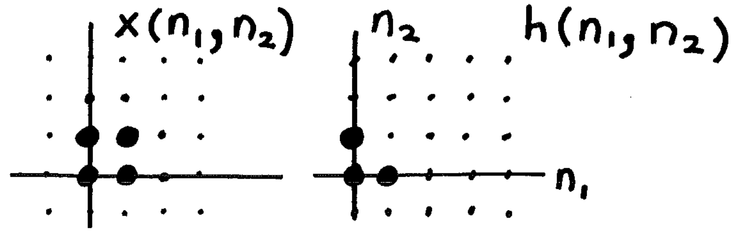
11/24/86 ; 3:30 to 4:25 p.m

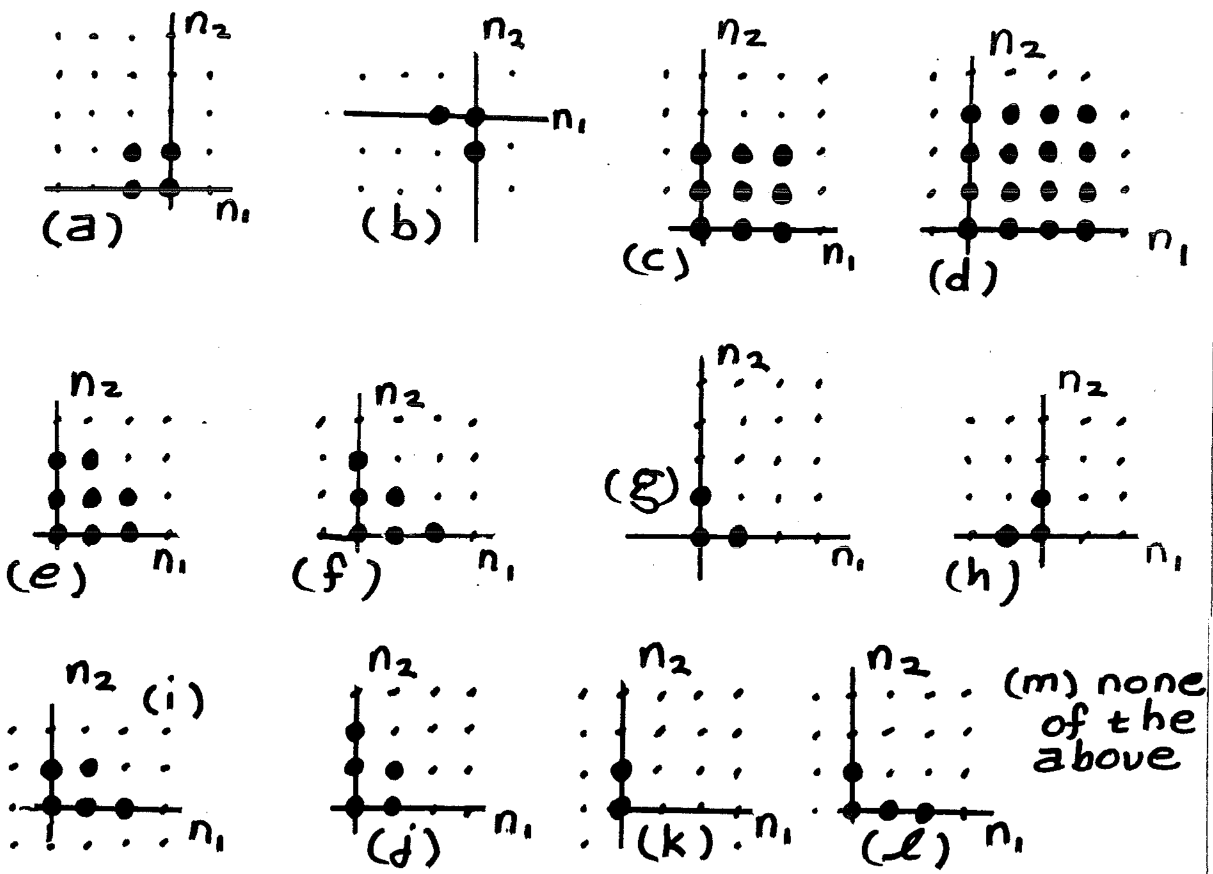
Score ______ /100

Information

1. Each problem is worth 25 points.
2. There is no penalty for guessing on multiple choice questions.
3. The exam is closed book and closed notes. You are allowed 1 page of notes, a calculator and one book of math tables and equations.
4. Do all of your work in this test booklet.

A big dot denotes a nonzero value and a small dot a zero value. Two functions, $x(n_1, n_2)$ and $h(n_1, n_2)$ are shown here:





Match one of the above to the expressions below.
($**$ = 2D convolution; $*_1$ = convolution in the n_1 direction.)

1. $x ** h$ ____
2. $x *_1 h$ ____
3. $x *_2 h$ ____
4. $h ** h$ ____
5. $h *_1 h$ ____
6. $h(-n_1, n_2)$ ...... ____
7. $h(-n_1, -n_2)$ .... ____
8. $x(n_1, n_2) \cdot h(-n_1, n_2)$ ____
9. $x \cdot h$ .... ____
10. $x ** x ** x$ .... ____

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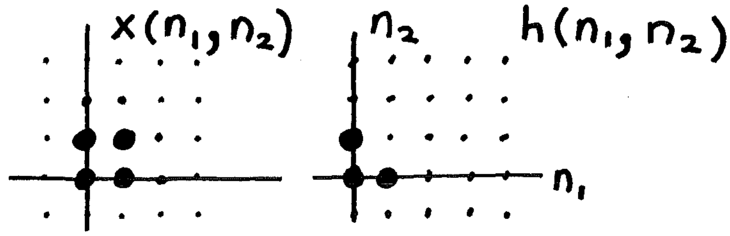

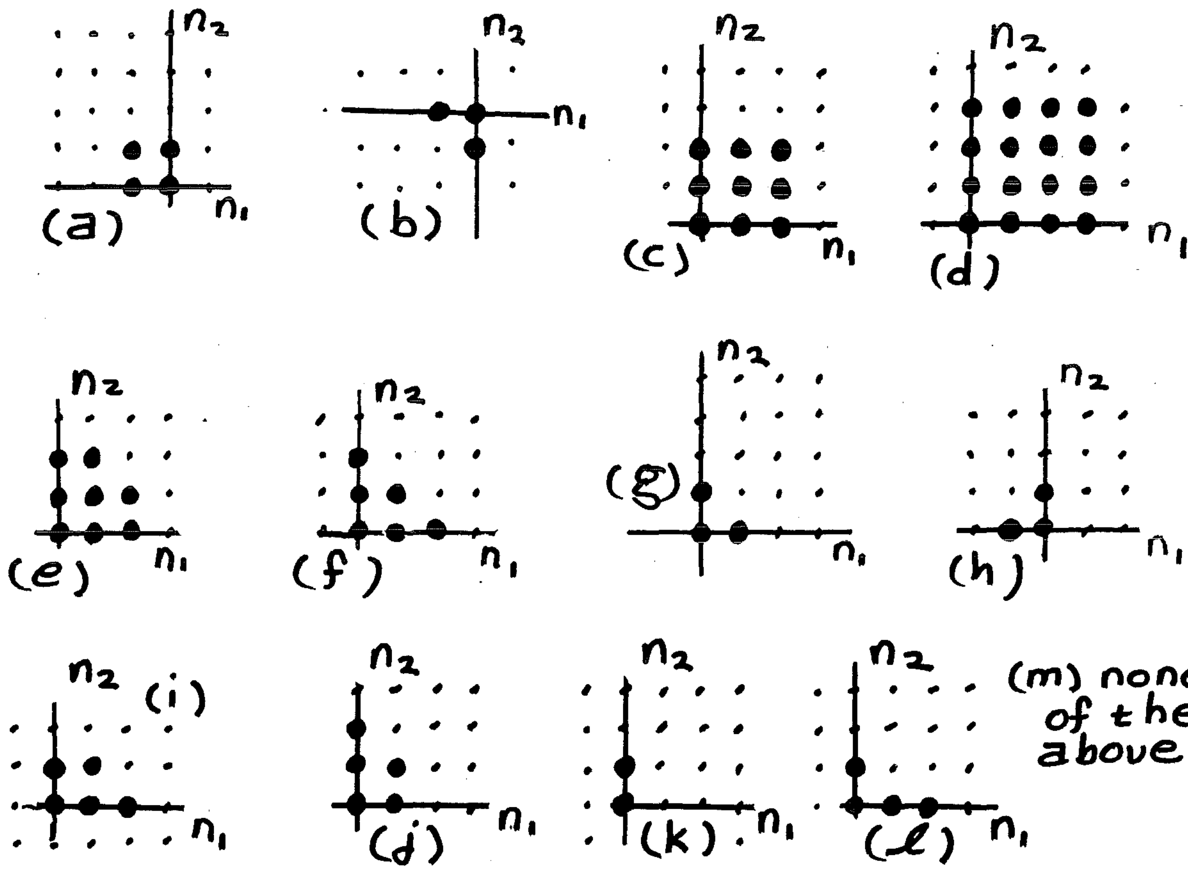

(m) none of the above

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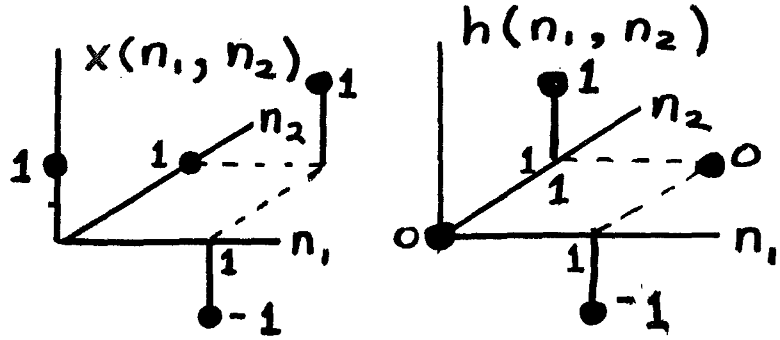
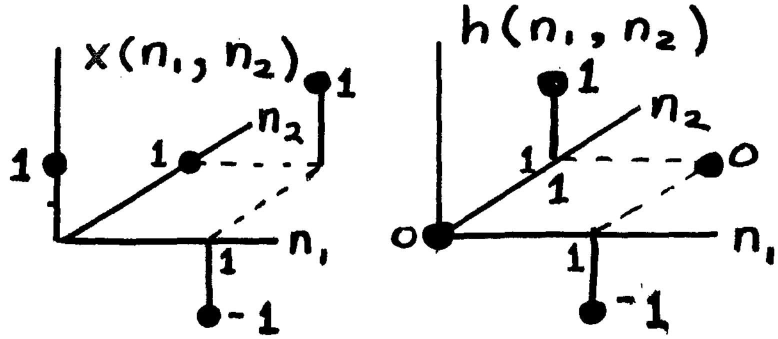


Compute and sketch the circular convolution of x with h.



Compute and sketch the circular convolution of x with h.

Compute the M-dimensional function, $W_M(r_M)$, whose continued projection to 1-D, gives the function:

$$W_1(t_1) = e^{-\pi t_1^2}$$

Hint : $\int_{-\infty}^{\infty} e^{-\pi a^2} \cos(2\pi a u)\,da = e^{-\pi u^2}$

An RA, notorious for goofing up data, samples a 2-D image whose spectrum is zero outside of a circle of known finite radius. Sampling was performed at the Nyquist density. The original image is gone, so we can't resample. What would be your reaction if:

(a) Every sample, except that at the origin, is zero. The value for the sample at the origin was lost.

(b) Same as (a), except that the value of the sample closest to the origin down the positive horizontal axis, was 1. The RA, though, says he isn't sure about that particular sample value.

Be specific as you can. Justify your answers.

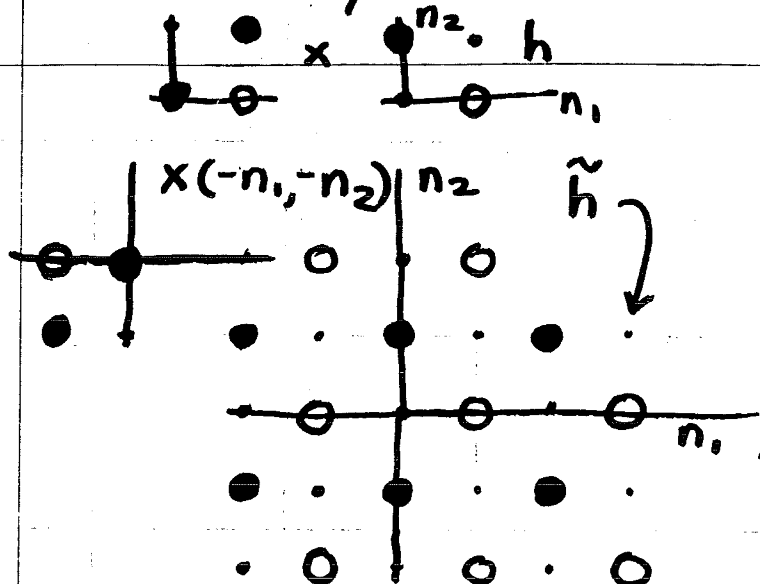
Scratch Sheet

EE595

Midterm #1 Solutions

1, e; 2. i; 3, j; 4, f; 5, ℓ; 6, h; 7, b; 8 k; 9 g; 10 d

2. Directly



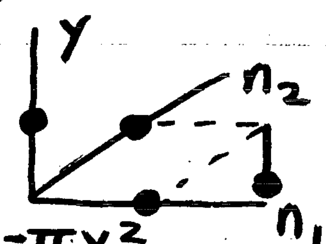
$y = x ** h$

$y(0,0) = 1$

$y(1,0) = 0$

$y(0,1) = 0$

$y(1,1) = -1$



3. Since $\int_{-\infty}^{\infty} e^{-\pi r^2} dx = e^{-\pi y^2}$

(Use hint with $u=0$). Thus:

$$w_M(r_M) = e^{-\pi r_M^2}$$

4. (a) The sample at the origin must also be zero. This follows from the restoration formula's linearity.

(b) If we lost 2 samples, and those remaining were zero, then the two lost samples would be zero also. Thus, the value of 1 is wrong. It should be zero.

name

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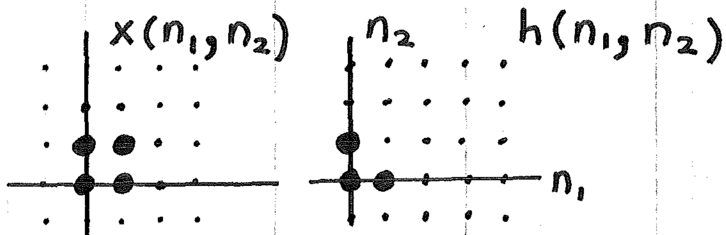

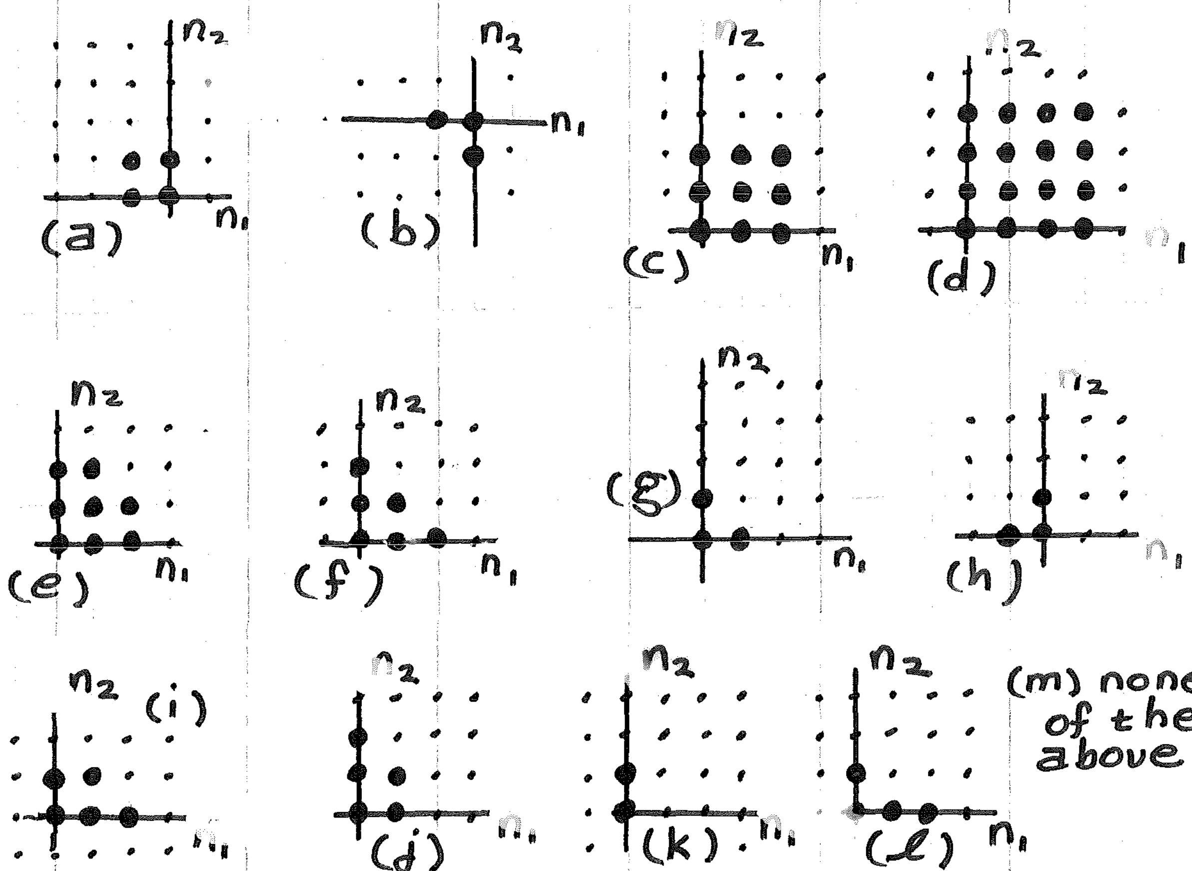

(m) none of the above

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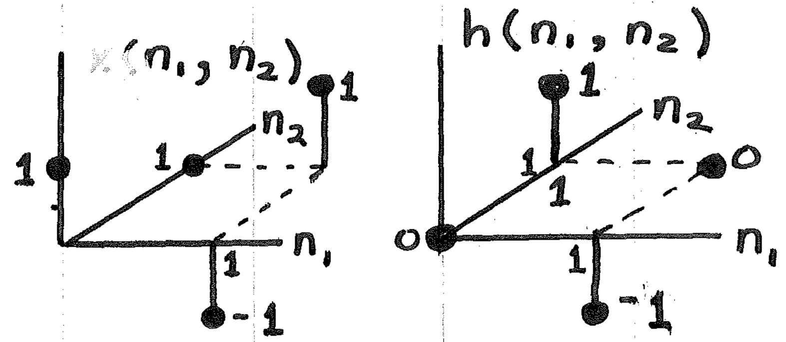

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(a) Every sample, except that at the origin, is zero. The value for the sample at the origin was lost.

(b) Same as (a), except that the value of the sample closest to the origin down the positive horizontal axis, was 1. The RA, though, says he isn't sure about that particular sample value.

Be specific as you can. Justify your answers.

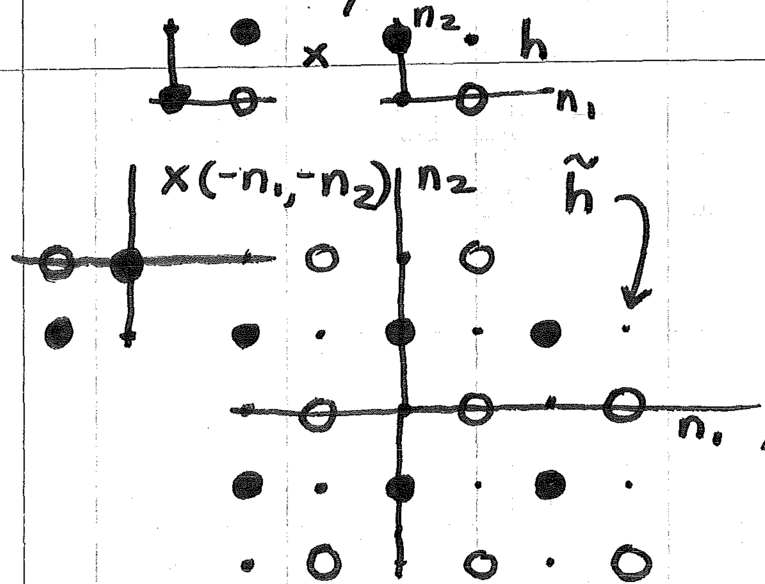
Scratch Sheet

EE595

Midterm #1 Solutions

1. e; 2. i; 3. j; 4. f; 5. l; 6. h; 7. b; 8 k; 9 g; 10 d

2. Directly



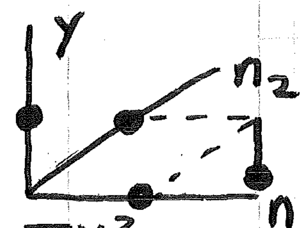
$y = x ** h$

$y(0,0) = 1$

$y(1,0) = 0$

$y(0,1) = 0$

$y(1,1) = -1$



3. Since $\int_{-\infty}^{\infty} e^{-\pi r^2} dx = e^{-\pi y^2}$

(Use hint with $u=0$). Thus:

$$w_M(r_M) = e^{-\pi r_M^2}$$

4. (a) The sample at the origin must also be zero. This follows from the restoration formula's linearity.

(b) If we lost 2 samples, and those remaining were zero, then the two lost samples would be zero also. Thus, the value of 1 is wrong. It should be zero.

Restoring Lost Samples

sampling theorem: $x_a(\vec{t}) = \sum_{\vec{n}} x[\vec{n}]\, f(\vec{t} - \underline{V}\vec{n})$; $x[\vec{n}] = x_a[\underline{V}$ (1)

We can use various f's. Define B as a periodic cell boundry (e.g. B = hexagon for hexogonally sampled signal) and C = spectrums region of support. Then, assuming $B \neq C$, two possible choices for the interpolating function are

$$f(\vec{t}) = \frac{|\det \underline{V}|}{(2\pi)^N} \int_D e^{j\vec{\Omega}^T\vec{t}}\, d\vec{\Omega}\ ;\ D = B \underline{\text{or}}\ C \qquad (2)$$

In some cases, $B = C$.

Let $\mathcal{M}$ denote a set of M N-dimensional vectors corresponding to the coordinates of M lost samples. We can write (1) as

$$x_a(\vec{t}) = \left[\sum_{\vec{n}\in\mathcal{M}} + \sum_{\vec{n}\notin\mathcal{M}}\right] x[\vec{n}]\, f(\vec{t} - \underline{V}\vec{n}) \qquad (3)$$

Recall $x[\vec{n}] = x_a(\underline{V}\ \vec{n})$. We sample (3) at the locations of the M lost samples:

$$x_a(\underline{V}\vec{k}) = \left[\sum_{\vec{n}\in\mathcal{M}} + \sum_{\vec{n}\notin\mathcal{M}}\right] x_a(\underline{V}\vec{n}) f(\underline{V}(\vec{k} - \vec{n}))\ ;\ \vec{k}\in\mathcal{M} \qquad (4)$$

Thus:

$$\sum_{\vec{n}\in\mathcal{M}} \left[\delta(\vec{n} - \vec{k}) - f(\underline{V}(\vec{k} - \vec{n}))\right] x_a(\underline{V}\vec{n}) = g(\vec{k})\ ;\ \vec{k}\in\mathcal{M} \qquad (5)$$

where

$$g(\vec{k}) = \sum_{\vec{n}\notin\mathcal{M}} x_a(\underline{V}\vec{n})\, f(\underline{V}(\vec{k} - \vec{n}))\ ;\ \vec{k}\in\mathcal{M} \qquad (6)$$

corresponds to M numbers that can be found from the known samples. The M unknown samples can be evaluated in (5) by solving M equations and M unknowns:

$$\begin{array}{c c} & \vec{n}_1 \quad \vec{n}_2 \quad \cdots \quad \vec{n}_M \\ \begin{array}{c}\vec{k}_1 \\ \vec{k}_2 \\ \vdots \\ \vec{k}_M\end{array} & \left[\begin{array}{c} \\ \delta(\vec{n} - \vec{k}) - f(\underline{V}(\vec{k} - \vec{n})) \\ \\ \end{array}\right] \end{array} \left[\begin{array}{c} \\ x_a(\underline{V}\vec{n}) \\ \\ \end{array}\right] = \left[\begin{array}{c} \\ g(\vec{k}) \\ \\ \end{array}\right] \qquad (7)$$

Notes:

1. A condition for solution of (5) is that the $M \times M$ matrix in (7) is not singular. Our conjecture is the matrix <u>is</u> singular when B (the cell boundry) is used in (2) and that it is <u>not</u> singular when $C \neq B \in C$.

2. For dimensions ≥ 2, there exist cases where an N dimensional signal can be sampled at a minimum density, yet still result in samples that are linearly dependent. e.g. A circular support for a spectrum requires hexogonal sampling for minimum density. Yet M samples can be restored if lost.

3. Our theory says if we loose* 10^{98} samples, we can restore them all. Note, however, the assumption that we have an infinite number of remaining samples each known to infinite precision. In practice, the finite number of known samples (truncation error) and data noise (e.g. round off error) will degade restoration, as will an increase in M. For the 1-D case, see Marks and Radbel, <u>IEEE</u> <u>Trans</u> <u>ASSP</u>, June, 1984.

4. For a single lost sample at the origin, (5) can be solved:

$$x_a(\vec{o}) = \frac{\sum_{\vec{n} \neq o} x_a(\underline{V}\vec{n})\, f(-\underline{V}\vec{n})}{1 - f(\vec{o})} \tag{8}$$

From (2):

$$f(\vec{o}) = \frac{|\det \underline{V}|}{(2\pi)^M} \int_D d\vec{\Omega} \tag{9}$$

Thus, for (8) to be valid:

$$\int_D d\vec{\Omega} \neq \frac{(2\pi)^M}{|\det \underline{V}|} = |\det \underline{U}| \tag{10}$$

This is an equality if $D = B$. (elaborate)

---

*10^{98} > number of atoms in the universe

EE595 MIDTERM Solutions

Due at beginning of class, ..., ..., ..

Staple your work. Use this as cover sheet. Neatness counts!

Any non-human source (except Bob Marks) is okay

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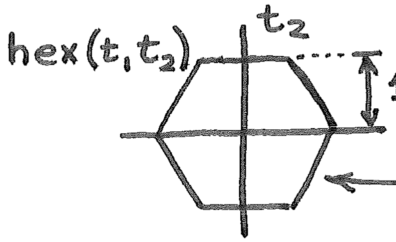
1. Work prob. # 1.20, p.55, in M dimensions
2. Define hex(t_1,t_2) as 1 inside & 0 outside:



Let the 2-D Fourier transform be hinc(Ω_1,Ω_2)

(a) Compute hinc (Ω_1,Ω_2)

(b) Evaluate and sketch $h(\Omega_1,0)$ and $h(0,\Omega_2)$

(c) What other 1-D slices of hinc (Ω_1,Ω_2) are equivalent to the slices in (b)?

(d) A 2-D filter has a frequency response

$$H(\omega_1,\omega_2) = \text{hex}\left(\frac{\omega_1}{W_1},\frac{\omega_2}{W_2}\right)$$

Compute the corresponing impulse response, $h[n_1 n_2]$

3. The Parzen window is the convolution of two triangle functions. A triangle is the convolution of two identical boxcar windows. The Parzen window is zero for $|t|>\tau$ and is unity at the origin. Extend this window to two dimensions using the rotated spectrum technique. (Evaluate $w_2(r)$ in closed form ie analytically - not digitally). Normalize. Plot on the same graph with a similarly normalized $w_1(t)$.

4. Work prob# 2.3, p.106

#1

1.20 $H(\vec{\omega}) = \sum_{\vec{n}} h[\vec{n}] e^{-j\vec{\omega}^T\vec{n}}$ ⇐ M dimensions

$$|H(\vec{\omega})|^2 = H(\vec{\omega})\, H^*(\vec{\omega})$$
$$= \sum_{\vec{n}} \sum_{\vec{m}} h[\vec{n}]\, h^*[\vec{m}]\, e^{-j\vec{\omega}^T(\vec{n}-\vec{m})}$$

$$\int_{\pi\!\!\pi} |H(\omega)|^2 d\omega = \sum_{\vec{n}} \sum_{\vec{m}} h[\vec{n}] h^*[\vec{m}] \int_{\pi\!\!\pi} e^{-j\vec{\omega}^T(\vec{n}-\vec{m})} d\omega$$

where $\int_{\pi\!\!\pi} = \underbrace{\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}}_{M \text{ terms}}$

Now:
$$\int_{\pi\!\!\pi} e^{-j\vec{\omega}^T\vec{k}} d\omega = \int_{-\pi}^{\pi} e^{-j\omega_1 k_1} d\omega_1 \cdots \int_{-\pi}^{\pi} e^{-j\omega_M k_M} d\omega_M$$
$$= (2\pi)^M \delta[\vec{k}]$$

Thus

$$\frac{1}{(2\pi)^M}\int_{\pi\!\!\pi} |H(\vec{\omega})|^2 d\vec{\omega} = \sum_{\vec{n}} |h[\vec{n}]|^2$$

For $M = 3$

$$\frac{1}{(2\pi)^3}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |H(\omega_1 \omega_2 \omega_2)|^2 d\omega_1 d\omega_2 d\omega_2$$
$$= \sum_{n_1}\sum_{n_2}\sum_{n_3} |h[n_1 n_2 n_3]|^2$$

For $H =$ ball, this becomes

$$\frac{1}{(2\pi)^3}\, \frac{4}{3}\pi W^3 = \sum_{n_1}\sum_{n_2}\sum_{n_3} |h[n_1 n_2 n_3]|^2$$
$$= S$$

1.20 (cont) In M dimensions:

From Wozencraft & Jacobs *Principles of Communication Engineering*, pp. 355-357, the volume of an M dimensional sphere of radius ρ is $B_M \rho^M$ where

$$B_M = \begin{cases} 2^M (\pi)^{\frac{(M-1)}{2}} \dfrac{\left(\frac{M-1}{2}\right)!}{M!} & ; M \text{ odd} \\ \pi^{M/2} / \left(\frac{M}{2}\right)! & ; M \text{ even} \end{cases}$$

Thus:

$$\sum_{\vec{n}} |h[\vec{n}]|^2 = \frac{1}{(2\pi)^M} \int_{FF} |H(\vec{\omega})|^2 d\omega$$

$$= \frac{1}{(2\pi)^M} B_M W^M$$

$$= \begin{cases} \pi^{-\left(\frac{M+1}{2}\right)} \dfrac{\left(\frac{M-1}{2}\right)!}{M!} & ; M \text{ odd} \\ \dfrac{1}{2^M \pi^{M/2} (M/2)!} & ; M \text{ even} \end{cases}$$

3.

Equation for Parzen Window
(from SPECTRAL ANALYSIS & TIME SERIES by Priestly)

Parzen has suggested the following lag window:

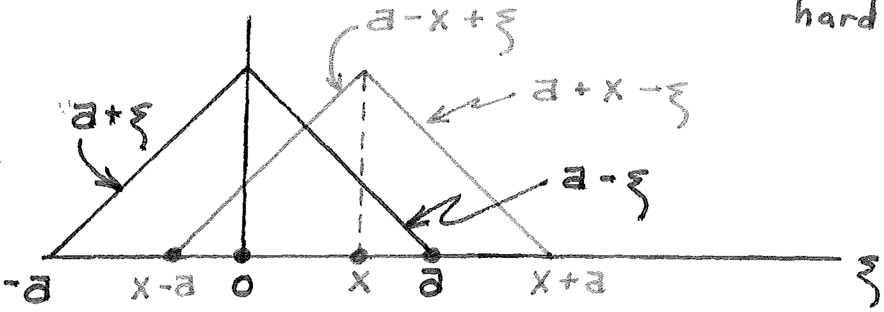
$$\lambda(s) = 1 - 6\frac{s^2}{M^2} + 6\frac{|s|^3}{M^3}, \quad |s| \le \frac{M}{2},$$
$$= 2\left(1 - \frac{|s|}{M}\right)^3, \quad \frac{M}{2} < |s| \le M,$$
$$= 0, \quad |s| > M. \tag{26}$$

For M even, the corresponding spectral window is given by

$$W(\theta) = \frac{6}{\pi M^3} \frac{\sin^4 \frac{M\theta}{4}}{\sin^4 \frac{\theta}{2}} \{1 - \frac{2}{3}\sin^2\frac{\theta}{2}\}. \tag{27}$$

The Parzen lag window may be derived by taking the Bartlett lag window (treated as a continuous function of s) and convolving it with itself. (The truncated periodogram lag window, the Bartlett lag window, and the Parzen lag window are related to the probability density functions of the sum of ,respectively, one, two, and three uniform $(-M, M)$ random variables.) Parzen sdf estimates, like the Bartlett and Daniell estimates, are always non-negative.

3. PARZEN WINDOW (worked out the hard way)



$$\frac{1}{c}W_1(x) = \int_{x-a}^{0}[a+\xi][(a-x)+\xi]\,d\xi + \int_0^x [a-\xi][(a-x)+\xi]\,d\xi + \int_x^a [a-\xi][(a+x)-\xi]\,d\xi$$

$$= \int_{x-a}^{0}[a(a-x)+(2a-x)\xi+\xi^2]\,d\xi$$

$$+ \int_0^x [a(a-x)+x\xi-\xi^2]\,d\xi$$

$$+ \int_x^a [a(a+x)-(2a+x)\xi+\xi^2]\,d\xi$$

$$= \left[a(a-x)\xi+\tfrac{1}{2}(2a-x)\xi^2+\tfrac{1}{3}\xi^3\right]_{x-a}^{0}$$

$$+\left[a(a-x)\xi+\tfrac{1}{2}x\xi^2-\tfrac{1}{3}\xi^3\right]_0^x$$

$$+\left[a(a+x)\xi-\tfrac{1}{2}(2a+x)\xi^2+\tfrac{1}{3}\xi^3\right]_x^a$$

$$= -a(a-x)(x-a) - \tfrac{1}{2}(2a-x)(x-a)^2-\tfrac{1}{3}(x-a)^3$$

$$+a(a-x)x+\tfrac{1}{2}x^3-\tfrac{1}{3}x^3$$

$$+a^2(a+x)-\tfrac{1}{2}(2a+x)a^2+\tfrac{1}{3}a^3$$

$$-a(a+x)x+\tfrac{1}{2}(2a+x)x^2-\tfrac{1}{3}x^3$$

B

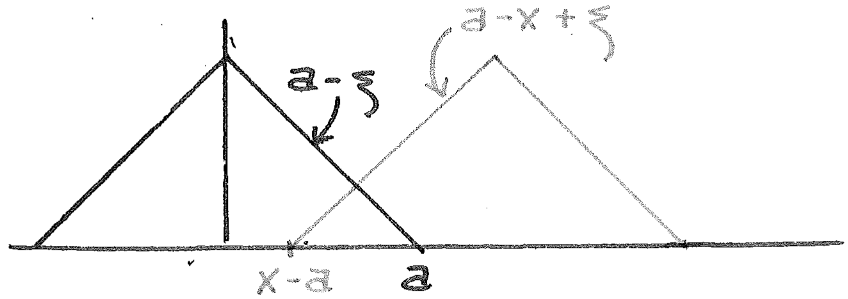
$$\frac{w_1(x)}{C} = a(x^2-2ax+a^2) + \left(\frac{x}{2}-a\right)(x^2-2ax+a^2)$$
$$-\frac{1}{3}(x^3-3x^2a+3xa^2-a^3)$$
$$+a(a-x)x+\frac{1}{6}x^3$$
$$+a^2(a+x)-\left(a+\frac{x}{2}\right)a^2+\frac{1}{3}a^3$$
$$-a(a+x)x+\left(a+\frac{x}{2}\right)x^2-\frac{1}{3}x^3$$

$$= a^3\left[\frac{1}{3}+1-1+\frac{1}{3}\right]$$
$$+a^2x\left[\frac{1}{2}-1+1+1-\frac{1}{2}-1\right]$$
$$+ax^2\left[-1+1-1-1+1\right]$$
$$+x^3\left[\frac{1}{2}-\frac{1}{3}+\frac{1}{6}+\frac{1}{2}-\frac{1}{3}\right]$$

$$= \frac{2}{3}a^3 - ax^2 + \frac{1}{2}x^3 \qquad (2)$$

Note:

$$\frac{w_1(0)}{C} = \frac{1}{C} = \frac{2a^3}{3} \Rightarrow C = \frac{3}{2a^3} \qquad (3)$$



$a \le x \le 2a$

$$\frac{w_1(x)}{c} = \int_{x-a}^{a} (a-\xi)\left[(a-x)+\xi\right]d\xi$$

$$= \int_{x-a}^{a}\left[a(a-x) + x\xi - \xi^2\right]d\xi$$

$$= \left[a(a-x)\xi + \tfrac{1}{2}x\xi^2 - \tfrac{1}{3}\xi^3\right]_{x-a}^{a}$$

$$= a^2(a-x) + \tfrac{1}{2}xa^2 - \tfrac{1}{3}a^3$$
$$- a(a-x)(x-a) - \tfrac{1}{2}x(x-a)^2 + \tfrac{1}{3}(x-a)^3$$

$$= a^2(a-x) + \tfrac{1}{2}xa^2 - \tfrac{1}{3}a^3$$
$$+ a(x^2 - 2ax + a^2) - \tfrac{1}{2}x(x^2 - 2ax + a^2)$$
$$+ \tfrac{1}{3}(x^3 - 3x^2a + 3xa^2 - a^3)$$

$$= a^3\left[1 - \tfrac{1}{3} + 1 - \tfrac{1}{3}\right]$$
$$+ a^2x\left[\cancel{-1} + \cancel{\tfrac{1}{2}} - 2 - \cancel{\tfrac{1}{2}} + \cancel{1}\right]$$
$$+ ax^2\left[1 + \cancel{1 - 1}\right]$$
$$+ x^3\left[-\tfrac{1}{2} + \tfrac{1}{3}\right]$$

$$= -\tfrac{1}{6}x^3 + ax^2 - 2a^2x + \tfrac{4}{3}a^3 \qquad (4)$$

Note:

$$\frac{w_1(2a)}{c} = -\tfrac{1}{6}(2a)^3 + a(2a)^2 - 2a^2(2a) + \tfrac{4}{3}a^3$$

$$= a^3\left[-\tfrac{8}{6} + 4 - 4 + \tfrac{4}{3}\right] = 0 \qquad (5)$$

Also, for $a < x < 2a$

$$\frac{1}{c}\frac{dw_1(x)}{dx} = -\frac{1}{2}x^2 + 2ax - 2a^2$$

Thus

$$\frac{1}{c}\frac{dw_1(2a)}{dx} = -\frac{1}{2}(2a)^2 + (2a)^2 - 2a^2$$

$$= [-2 + 4 - 2]\,a^2 = 0 \qquad (6)$$

Good!

Note: From part 1:

$$\frac{w_1(a^-)}{c} = \left[\frac{2}{3} - 1 + \frac{1}{2}\right] a^3 = \frac{1}{6} a^3 \qquad (7)$$

From part 2:

$$\frac{w_1(a^+)}{c} = \left[-\frac{1}{6} + 1 - 2 + \frac{4}{3}\right] a^3 = \frac{1}{6} a^3 \qquad (8)$$

$\therefore$ We get continuity @ a.

In summary:

$$w_1(x) = \begin{cases} \frac{3}{2a^3}\left[\frac{1}{2}x^3 - ax^2 + \frac{2}{3}a^3\right] & ; 0 \le x \le a \\ \frac{3}{2a^3}\left[-\frac{x^3}{6} + ax^2 - 2a^2x + \frac{4a^3}{3}\right] & ; a \le x \le 2a \\ 0 & ; x \ge 2a \\ w_1(-x) & ; x \le 0 \end{cases} \qquad (9)$$

Or, since $\tau = 2a$:

$$w_1(x) = \begin{cases} \frac{3}{2a^3}\left[\frac{1}{2}x^3 - \frac{x^2\tau}{2} + \frac{2}{3}\cdot\frac{\tau^3}{8}\right] & ; 0 \le x \le \tau/2 \\ \frac{3}{2a^3}\left[-\frac{x^3}{6} + \frac{x^2\tau}{2} - \frac{\tau^2}{2}x + \frac{4\tau^3}{3\cdot 8}\right] & ; \frac{\tau}{2} \le x \le \tau \end{cases} \qquad (10)$$

$$= \begin{cases} \frac{3}{4a^3}\left[x^3 - \tau x^2 + \frac{1}{6}\tau^3\right] & ; 0 \le x \le \tau/2 \\ \frac{-1}{4a^3}\left[x^3 - 3\tau x^2 + 3\tau^2 x - \tau^3\right] & ; \frac{\tau}{2} \le x \le \tau \end{cases} \qquad (11)$$

$$= \begin{cases} \frac{3}{4a^3}\left[x^2(x-\tau) + \frac{1}{6}\tau^3\right] & ; 0 \le x \le \tau/2 \\ \frac{1}{4a^3}(\tau - x)^3 & ; \frac{\tau}{2} \le x \le \tau \end{cases} \qquad (12)$$

$$= \begin{cases} \frac{6}{\tau^3}\left[x^2(x-\tau) + \frac{1}{6}\tau^3\right] & ; 0 \le x \le \tau/2 \\ \frac{2}{\tau^3}(\tau - x)^3 & ; \frac{\tau}{2} \le x \le \tau \end{cases} \qquad (13)$$

Note:

$$w_1\left(\tfrac{\tau}{2}-\right) = \frac{6}{\tau^3}\left[\frac{\tau^2}{4}\left(\frac{\tau}{2}-\tau\right)+\frac{\tau^3}{6}\right] \tag{14}$$

$$= 6\left[\left(\frac{1}{4}\right)\left(-\frac{1}{2}\right)+\frac{1}{6}\right] = 6\,\frac{4-3}{24} = \frac{1}{4}$$

and

$$w_1\left(\tfrac{\tau}{2}+\right) = \frac{2}{\tau^3}\left(\tau-\frac{\tau}{2}\right)^3 = 2\,\frac{1}{8} = \frac{1}{4} \tag{15}$$

$\therefore$ Continuity @ $\tau/2$ Good!

Now, from (13)

$$\frac{dw_1}{dx} = \begin{cases} \dfrac{6}{\tau^3}\left[3x^2 - 2\tau x\right] & ;\ 0 < x < \tau/2 \\[2ex] \dfrac{-6}{\tau^3}(x-\tau)^2 & ;\ \dfrac{\tau}{2} < x < \tau \end{cases} \tag{16}$$

Continuous @ $\frac{\tau}{2}$?

$$\frac{dw_1(\tau/2-)}{dx} = \frac{6}{\tau^3}\left[\frac{3\tau^2}{\cancel{8}4} - \tau^2\right] = 6\times\left(-\frac{1}{4\tau}\right) = \frac{-3}{2\tau} \tag{17}$$

$$\frac{dw_1(\tau/2+)}{dx} = \frac{-6}{\tau^3}(x-\tau)^2\Big|_{x=\frac{\tau}{2}} = \frac{-6}{\tau}\times\frac{1}{4} = \frac{-3}{2\tau} \tag{18}$$

YES!

Computing Inverse Abel

$$W_2(r) = -\frac{1}{\pi}\int_r^{\tau} \sqrt{x^2 - r^2}\ \frac{d}{dx}\frac{1}{x}\frac{dw_1(x)}{dx}\ dx \qquad (19)$$

From (16)

$$\frac{1}{x}\frac{dw_1}{dx} = \begin{cases} \frac{6}{\tau^3}\left[3x - 2\tau\right] = \frac{6}{\tau^3}\left[3x - 2\tau\right] & ; 0 \le x \le \tau/2 \\ -\frac{6}{\tau^3}\frac{x^2 - 2\tau x + \tau^2}{x} = \frac{-6}{\tau^3}\left[x - 2\tau + \frac{\tau^2}{x}\right] & ; \frac{\tau}{2} < x < \tau \end{cases} \qquad (20)$$

and

$$\frac{d}{dx}\frac{1}{x}\frac{dw_1(x)}{dx} = \begin{cases} \frac{18}{\tau^3} & ; 0 \le x \le \tau/2 \\ -\frac{6}{\tau^3}\left[1 - \frac{\tau^2}{x^2}\right] = \frac{6}{\tau^3}\left[\frac{\tau^2}{x^2} - 1\right] & ; \frac{\tau}{2} < x < \tau \end{cases} \qquad (21)$$

- For $\frac{\tau}{2} < r < \tau$

$$W_2(r) = -\frac{1}{\pi}\int_r^{\tau} \sqrt{x^2 - r^2}\ \frac{6}{\tau^3}\left[\frac{\tau^2}{x^2} - 1\right] dx \qquad (22)$$

Useful Integrals:

CRC, p.410 #178:

$$\int \frac{\sqrt{x^2 - r^2}}{x^2}\,dx = -\frac{\sqrt{x^2 - r^2}}{x} + \ln\left(x + \sqrt{x^2 - r^2}\right) \qquad (23)$$

CRC p 408 #156

$$\int \sqrt{x^2 - r^2}\,dx = \frac{1}{2}\left[x\sqrt{x^2 - r^2} - r^2 \ln\left(x + \sqrt{x^2 - r^2}\right)\right] \qquad (24)$$

Thus, (22) becomes:

$$W_2(r) = \frac{-6}{\pi\tau^3}\left[\tau^2\left\{\frac{-\sqrt{x^2-r^2}}{x} + \ln\left(x+\sqrt{x^2-r^2}\right)\right\}\Big|_r^\tau\right.$$

$$\left. - \frac{1}{2}\left\{x\sqrt{x^2-r^2} - r^2\ln\left(x+\sqrt{x^2-r^2}\right)\right\}\Big|_r^\tau\right]$$

$$= \frac{-6}{\pi\tau^3}\left[\tau^2\left\{\frac{-\sqrt{\tau^2-r^2}}{\tau} + \ln\left(\frac{\tau+\sqrt{\tau^2-r^2}}{r}\right)\right\}\right.$$

$$\left. - \frac{1}{2}\left\{\tau\sqrt{\tau^2-r^2} - r^2\ln\frac{\tau+\sqrt{\tau^2-r^2}}{r}\right\}\right]$$

$$= \frac{-6}{\pi\tau^2}\left[\sqrt{\tau^2-r^2}\left[-\tau-\frac{\tau}{2}\right]\right.$$

$$\left. + \left(\tau^2+\frac{r^2}{2}\right)\ln\left(\frac{\tau+\sqrt{\tau^2-r^2}}{r}\right)\right]$$

$$= \frac{6}{\pi\tau^2}\left[\frac{3\tau}{2}\sqrt{\tau^2-r^2} - \frac{1}{2}\left(2\tau^2+r^2\right)\ln\frac{\tau+\sqrt{\tau^2-r^2}}{r}\right]$$

$$= \frac{3}{\pi\tau^2}\left[3\tau\sqrt{\tau^2-r^2} - \left(2\tau^2+r^2\right)\ln\frac{\tau+\sqrt{\tau^2-r^2}}{r}\right] \tag{25}$$

For $0 < r < \frac{\tau}{2}$

$$w_2(r) = -\frac{1}{\pi}\left[\int_r^{\tau/2} + \int_{\tau/2}^{\tau}\right] \sqrt{x^2-r^2}\,\frac{d}{dx}\frac{1}{x}\frac{dw_1}{dx}\,dx$$

$$= w_2\left(\tfrac{\tau}{2}\right) - \frac{1}{\pi}\int_r^{\tau/2}\sqrt{x^2-r^2}\,\frac{d}{dx}\frac{1}{x}\frac{dw_1}{dx}\,dx \quad (26)$$

where $w_2(\frac{\tau}{2})$ is computed from (25). Using (21):

$$w_2(r) = w_2\left(\tfrac{\tau}{2}\right) - \frac{18}{\pi\tau^3}\int_r^{\tau/2}\sqrt{x^2-r^2}\,dx \quad (27)$$

Using (24):

$$w_2(r) = w_2\left(\tfrac{\tau}{2}\right) - \frac{18}{\pi\tau^3}\,\frac{1}{2}\left[x\sqrt{x^2-r^2} - r^2\ln\left(x+\sqrt{x^2-r^2}\right)\right]_r^{\tau/2}$$

$$= w_2\left(\tfrac{\tau}{2}\right) - \frac{9}{\pi\tau^3}\left[\frac{\tau}{2}\sqrt{\left(\tfrac{\tau}{2}\right)^2-r^2} - r^2\ln\left(\frac{\frac{\tau}{2}+\sqrt{\left(\frac{\tau}{2}\right)^2-r^2}}{r}\right)\right] \quad (28)$$

Combining (25) & (28):

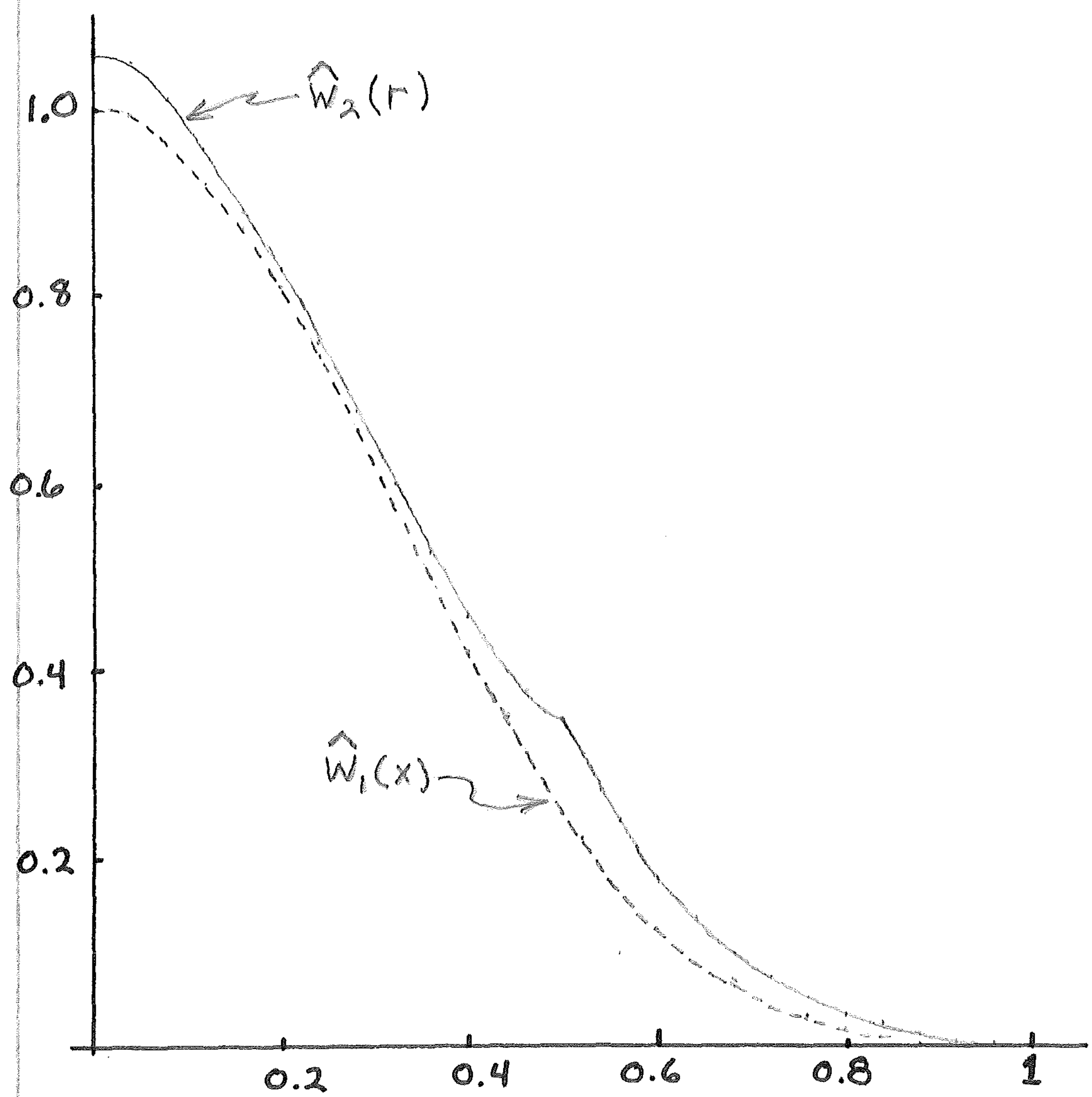
$$w_2(r) = \begin{cases} w_2\left(\tfrac{\tau}{2}\right) - \dfrac{9}{\pi\tau^3}\left[\dfrac{\tau}{2}\sqrt{\dfrac{\tau^2}{4}-r^2} - r^2\ln\left(\dfrac{\frac{\tau}{2}+\sqrt{\frac{\tau^2}{4}-r^2}}{r}\right)\right] & ;\ 0<r<\tau/2 \\[2ex] \dfrac{3}{\pi\tau^3}\left[3\tau\sqrt{\tau^2-r^2} - (2\tau^2+r^2)\ln\left(\dfrac{\tau+\sqrt{\tau^2-r^2}}{r}\right)\right] & \dfrac{\tau}{2}<r<\tau \end{cases} \quad (29)$$

checks!

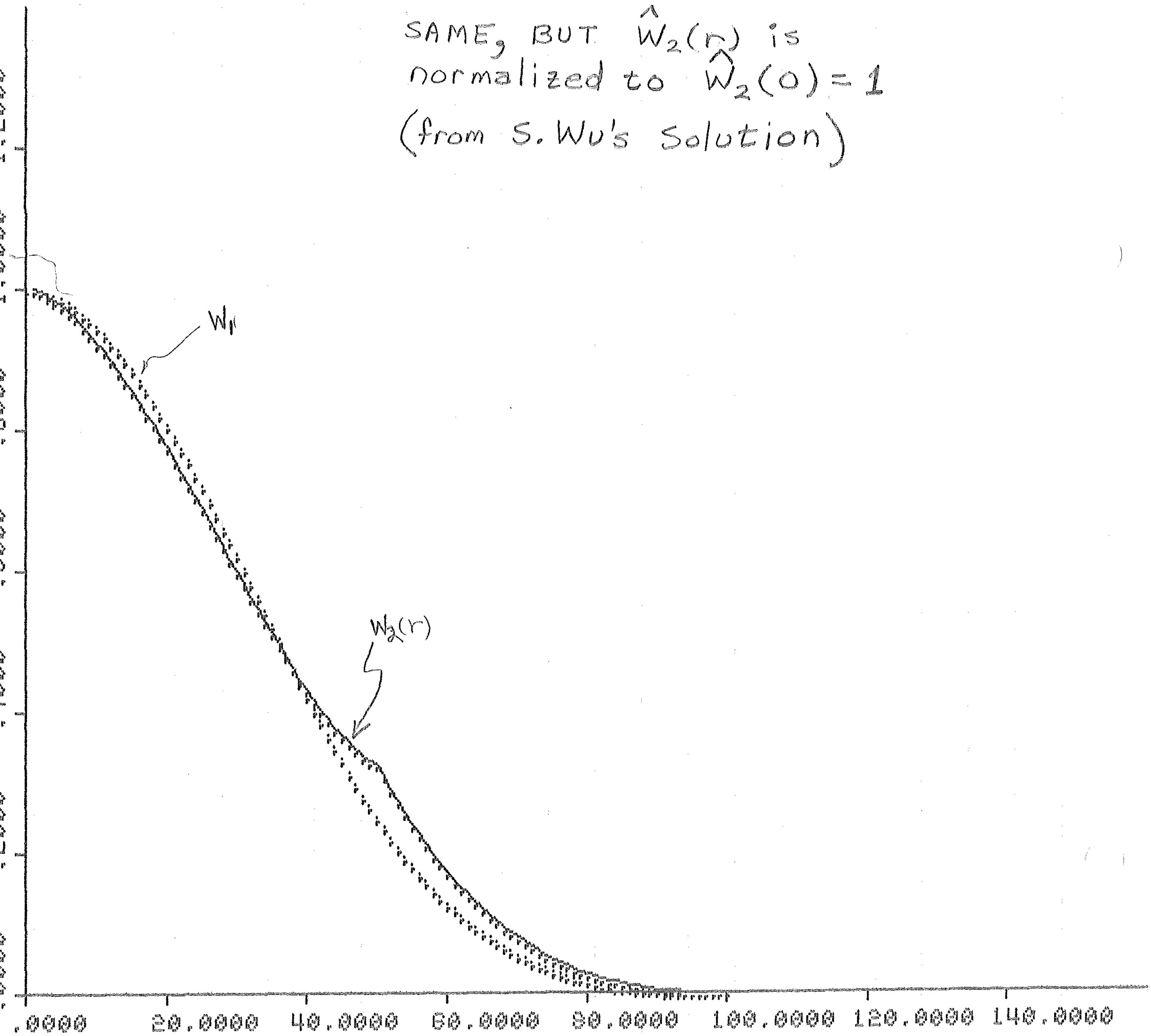
Normalizing:

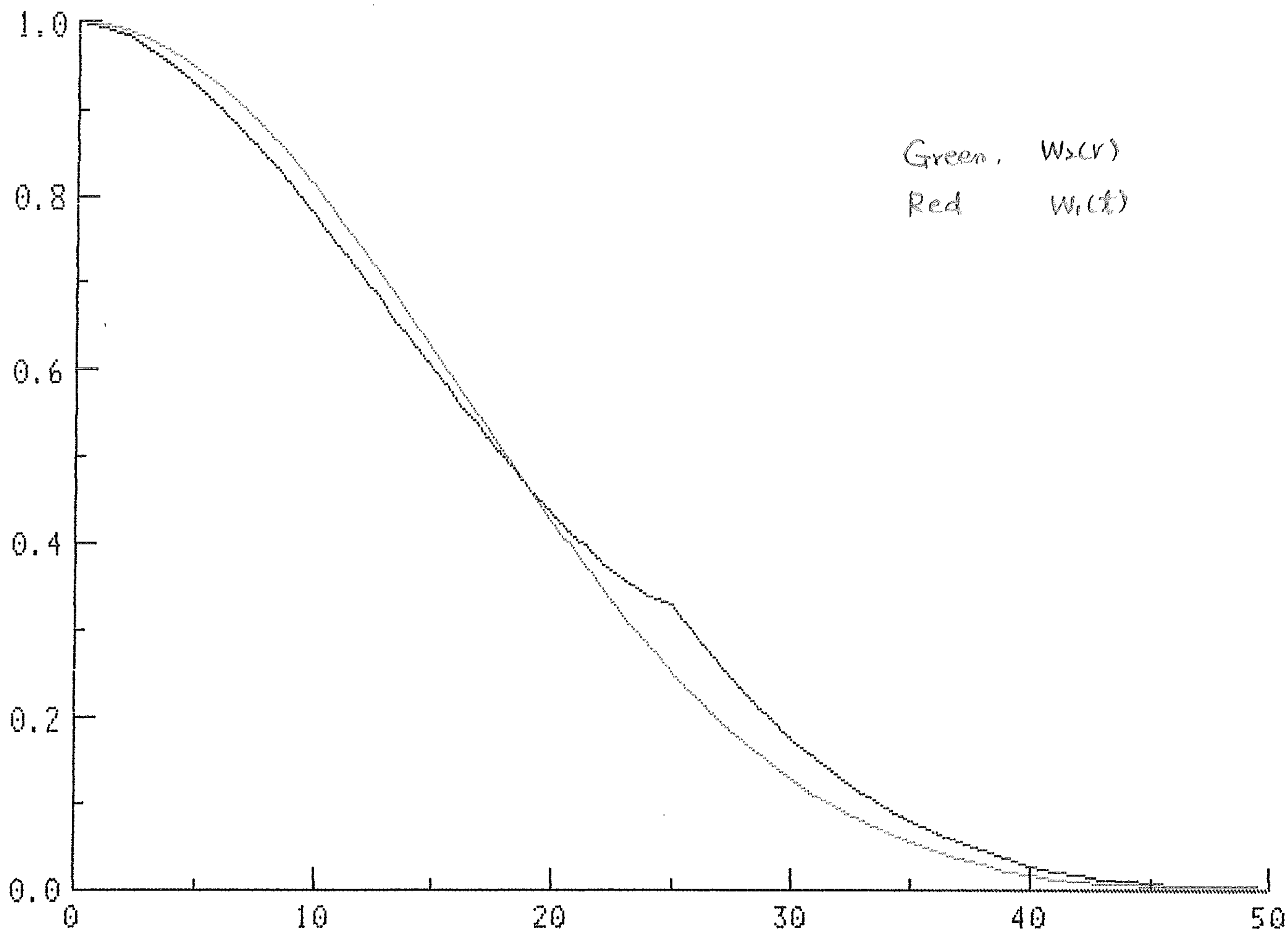
$$\tau\,w_2(r\tau) = \begin{cases} \tau w_2\left(\tfrac{r\tau}{2}\right) - \dfrac{9}{\pi}\left[\dfrac{1}{2}\sqrt{\dfrac{1}{4}-r^2} - r^2\ln\left(\dfrac{\frac{1}{2}+\sqrt{\frac{1}{4}-r^2}}{r}\right)\right] \\[2ex] \dfrac{3}{\pi}\left[3\sqrt{1-r^2} - (2+r^2)\ln\left(\dfrac{1+\sqrt{1-r^2}}{r}\right)\right] \end{cases}$$

$$= \hat{w}_2(r) = \begin{cases} \hat{w}_2\left(\tfrac{1}{2}\right) - \dfrac{9}{\pi}\left[\dfrac{1}{2}\sqrt{\dfrac{1}{4}-r^2} - r^2\ln\left(\dfrac{\frac{1}{2}+\sqrt{\frac{1}{4}-r^2}}{r}\right)\right] & ;\ 0<r<\tfrac{1}{2} \\[2ex] \dfrac{3}{\pi}\left[3\sqrt{1-r^2} - (2+r^2)\ln\dfrac{1+\sqrt{1-r^2}}{r}\right. & ;\ \tfrac{1}{2}<r<1 \end{cases}$$



$\hat{W}_2(r)$
$\hat{W}_1(x)$
1.0
0.8
0.6
0.4
0.2
0.2
0.4
0.6
0.8
1


SAME, BUT $\hat{W}_2(r)$ is
normalized to $\hat{W}_2(0) = 1$
(from S. Wu's Solution)
W_1
$W_2(r)$
1.2000
1.0000
.8000
.6000
.4000
.2000
.0000
20.0000
40.0000
60.0000
80.0000
100.0000
120.0000
140.0000

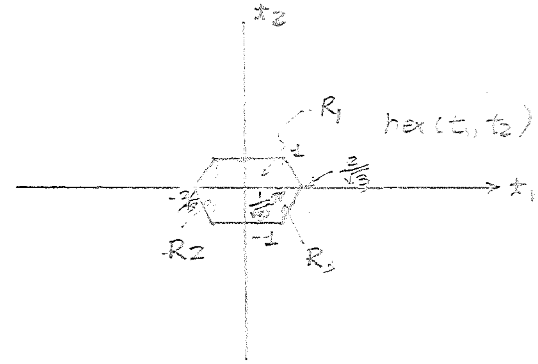


From Work of T. Ku.

2

SOLUTION OF K.F Cheung

2. Define



a) $\mathrm{hinc}(\Omega_1, \Omega_2) = \mathcal{F}[hex(t_1, t_2)]$

$$= \mathcal{F}[hex(t_1,t_2)]_{(t_1,t_2)\in R_1} + \mathcal{F}[hex(t_1,t_2)]_{(t_1,t_2)\in R_2} + \mathcal{F}[hex(t_1,t_2)]_{(t_1,t_2)\in R_3} \quad (1)$$

$$\mathcal{F}[hex(t_1,t_2)]_{(t_1,t_2)\in R_1} = \int_{-1}^{1}\int_{-1/\sqrt{3}}^{1/\sqrt{3}} e^{-j\Omega_1 t_1} e^{-j\Omega_2 t_2}\, dt_1\, dt_2$$

$$= \int_{-1}^{1} e^{-j\Omega_2 t_2}\, dt_2 \int_{\frac{1}{\sqrt{3}}}^{1/\sqrt{3}} e^{-j\Omega_1 t_1}\, dt_1$$

$$= 4\,\frac{\sin\Omega_2}{\Omega_2}\,\frac{\sin\Omega_1/\sqrt{3}}{\Omega_1} \quad (2)$$

$$\mathcal{F}[hex(t_1,t_2)]_{(t_1,t_2)\in R_3} = \int_{1/\sqrt{3}}^{2/\sqrt{3}}\int_{-(2-\sqrt{3}t_1)}^{2-\sqrt{3}t_1} e^{-j\Omega_1 t_1} e^{-j\Omega_2 t_2}\, dt_1\, dt_2$$

$$= \int_{1/\sqrt{3}}^{2/\sqrt{3}} e^{-j\Omega_1 t_1}\left[\int_{0}^{2-\sqrt{3}t_1} e^{-j\Omega_2 t_2} + e^{j\Omega_2 t_2}\, dt_2\right] dt_1$$

$$= \int_{1/\sqrt{3}}^{2/\sqrt{3}} e^{-j\Omega_1 t_1}\left[\int_{0}^{2-\sqrt{3}t_1} 2\cos\Omega_2 t_2\, dt_2\right] dt_1$$

$$= \int_{1/\sqrt{3}}^{2/\sqrt{3}} e^{-j\Omega_1 t_1}\cdot\frac{2}{\Omega_2}\sin\left[\Omega_2(2-\sqrt{3}t_1)\right] dt_1$$

$$= \frac{2}{\Omega_2}\int_{1/\sqrt{3}}^{2/\sqrt{3}} e^{-j\Omega_1 t_1}\left(\sin 2\Omega_2 \cos\sqrt{3}\Omega_2 t_1 - \sin\sqrt{3}\Omega_2 t_1 \cos 2\Omega_2\right) dt_1$$

$$= \frac{2}{\Omega_2}\left\{ \sin 2\Omega_2 \int_{1/\sqrt{3}}^{2/\sqrt{3}} e^{-j\Omega_1 t_1} \cos\sqrt{3}\Omega_2 t_1 \, dt_1 \right.$$

$$\left. - \cos 2\Omega_2 \int_{1/\sqrt{3}}^{2/\sqrt{3}} e^{-j\Omega_1 t_1} \sin\sqrt{3}\Omega_2 t_1 \, dt_1 \right\}$$

$$= \frac{2}{\Omega_2}\left\{ \frac{\sin 2\Omega_2}{2}\left[\frac{e^{j\frac{2}{\sqrt{3}}(\sqrt{3}\Omega_2-\Omega_1)} - e^{j\frac{1}{\sqrt{3}}(\sqrt{3}\Omega_2-\Omega_1)}}{j[\sqrt{3}\Omega_2-\Omega_1]} + \frac{e^{-j\frac{2}{\sqrt{3}}(\sqrt{3}\Omega_2+\Omega_1)} - e^{-j\frac{1}{\sqrt{3}}(\sqrt{3}\Omega_2+\Omega_1)}}{-j(\sqrt{3}\Omega_2+\Omega_1)} \right] \right.$$

$$\left. - \frac{\cos 2\Omega_2}{2j}\left[\frac{e^{j\frac{2}{\sqrt{3}}(\sqrt{3}\Omega_2-\Omega_1)} - e^{j\frac{1}{\sqrt{3}}(\sqrt{3}\Omega_2-\Omega_1)}}{j(\sqrt{3}\Omega_2-\Omega_1)} - \frac{e^{-j\frac{2}{\sqrt{3}}(\sqrt{3}\Omega_2+\Omega_1)} - e^{-j\frac{1}{\sqrt{3}}(\sqrt{3}\Omega_2+\Omega_1)}}{-j(\sqrt{3}\Omega_2+\Omega_1)} \right] \right\} \quad ③$$

$$\mathcal{F}[hex(t_1,t_2)]_{(t_1,t_2)\in R_2} = \int_{-2/\sqrt{3}}^{-1/\sqrt{3}} \int_{2-\sqrt{3}t_1}^{2+\sqrt{3}t_1} e^{-j\Omega_1 t_1} e^{-j\Omega_2 t_2} \, dt_1 \, dt_2$$

$$= \int_{1/\sqrt{3}}^{2/\sqrt{3}} \int_{-2+\sqrt{3}t_1}^{2-\sqrt{3}t_1} e^{j\Omega_1 t_1} e^{-j\Omega_2 t_2} \, dt_1 \, dt_2$$

$$= \frac{2}{\Omega_2}\left\{ \frac{\sin 2\Omega_2}{2}\left[\frac{e^{j\frac{2}{\sqrt{3}}(\sqrt{3}\Omega_2+\Omega_1)} - e^{j\frac{1}{\sqrt{3}}(\sqrt{3}\Omega_2+\Omega_1)}}{j(\sqrt{3}\Omega_2+\Omega_1)} + \frac{e^{-j\frac{2}{\sqrt{3}}(\sqrt{3}\Omega_2-\Omega_1)} - e^{-j\frac{1}{\sqrt{3}}(\sqrt{3}\Omega_2-\Omega_1)}}{-j(\sqrt{3}\Omega_2-\Omega_1)} \right] \right.$$

$$\left. - \frac{\cos 2\Omega_2}{2j}\left[\frac{e^{j\frac{2}{\sqrt{3}}(\sqrt{3}\Omega_2+\Omega_1)} - e^{j\frac{1}{\sqrt{3}}(\sqrt{3}\Omega_2+\Omega_1)}}{j(\sqrt{3}\Omega_2+\Omega_1)} - \frac{e^{-j\frac{2}{\sqrt{3}}(\sqrt{3}\Omega_2-\Omega_1)} - e^{-j\frac{1}{\sqrt{3}}(\sqrt{3}\Omega_2-\Omega_1)}}{-j(\sqrt{3}\Omega_2-\Omega_1)} \right] \right\} \quad ④$$

$$\therefore hinc(\Omega_1,\Omega_2) = ② + ③ + ④$$

$$= \frac{4\sin\Omega_2}{\Omega_2}\,\frac{\sin\Omega_1/\sqrt{3}}{\Omega_1} + \left\{ \frac{\sin 2\Omega_2}{\Omega_2}\left[\frac{2\sin\left(\frac{2}{\sqrt{3}}(\sqrt{3}\Omega_2-\Omega_1)\right) - 2\sin\left(\frac{1}{\sqrt{3}}(\sqrt{3}\Omega_2-\Omega_1)\right)}{(\sqrt{3}\Omega_2-\Omega_1)} \right.\right.$$

$$\left. + \frac{2\sin\left(\frac{2}{\sqrt{3}}(\sqrt{3}\Omega_2+\Omega_1)\right) - 2\sin\left(\frac{1}{\sqrt{3}}(\sqrt{3}\Omega_2+\Omega_1)\right)}{(\sqrt{3}\Omega_2+\Omega_1)} \right]$$

$$+ \frac{\cos 2\Omega_2}{\Omega_2}\left[\frac{2\cos\left(\frac{2}{\sqrt{3}}(\sqrt{3}\Omega_2-\Omega_1)\right) - 2\cos\left(\frac{1}{\sqrt{3}}(\sqrt{3}\Omega_2-\Omega_1)\right)}{(\sqrt{3}\Omega_2-\Omega_1)} + \frac{2\cos\left(\frac{2}{\sqrt{3}}(\sqrt{3}\Omega_2+\Omega_1)\right) - 2\cos\left(\frac{1}{\sqrt{3}}(\sqrt{3}\Omega_2+\Omega_1)\right)}{(\sqrt{3}\Omega_2+\Omega_1)} \right.$$

$$= 4\sin\frac{\Omega_2}{\Omega_2}\sin\frac{\Omega_1\sqrt{3}}{\Omega_1} + \frac{2\cos\left[2\Omega_1/\sqrt{3}\right]}{\Omega_2(\sqrt{3}\Omega_2-\Omega_1)} - \frac{2\cos\left[\Omega_2+\Omega_1/\sqrt{3}\right]}{\Omega_2(\sqrt{3}\Omega_2-\Omega_1)}$$

$$+ \frac{2\cos\left[2\Omega_1/\sqrt{3}\right]}{\Omega_2(\sqrt{3}\Omega_2+\Omega_1)} - \frac{2\cos\left[\Omega_2-\Omega_1/\sqrt{3}\right)}{\Omega_2\left[\sqrt{3}\Omega_2+\Omega_1\right]} \quad //$$

b) ~~find~~ $h(\Omega_1, 0) =$

for region R_1: $h(\Omega_1, 0)\big|_{R_1} = \dfrac{4\sin\Omega_1/\sqrt{3}}{\Omega_1}$

for region R_2: $h(\Omega_1, 0)\big|_{R_3} = \displaystyle\int_{1/\sqrt{3}}^{2/\sqrt{3}}\int_{-(2-\sqrt{3}t_1)}^{2-\sqrt{3}t_1} e^{-j\Omega_1 t_1}\, dt_2\, dt_1$

$$= \int_{1/\sqrt{3}}^{2/\sqrt{3}} 2(2-\sqrt{3}t_1)\, e^{-j\Omega_1 t_1}\, dt_1$$

$$= 4\int_{1/\sqrt{3}}^{2/\sqrt{3}} e^{-j\Omega_1 t_1}\, dt_1 - 2\sqrt{3}\int_{1/\sqrt{3}}^{2/\sqrt{3}} t_1 e^{-j\Omega_1 t_1}\, dt_1$$

$$= 4\,\frac{e^{-j\frac{2}{\sqrt{3}}\Omega_1} - e^{-j\frac{1}{\sqrt{3}}\Omega_1}}{-j\Omega_1} - 2\sqrt{3}\left[\frac{\frac{2}{\sqrt{3}}e^{-j\frac{2}{\sqrt{3}}\Omega_1} - \frac{1}{\sqrt{3}}e^{-j\frac{1}{\sqrt{3}}\Omega_1}}{-j\Omega_1} + \frac{e^{-j\frac{2}{\sqrt{3}}\Omega_1} - e^{-j\frac{1}{\sqrt{3}}\Omega_1}}{\Omega_1^2}\right]$$

$h(\Omega_1, 0)\big|_{R_2} = h(-\Omega_1, 0)_{R_3}$

$$= 4\,\frac{e^{j\frac{2}{\sqrt{3}}\Omega_1} - e^{j\frac{1}{\sqrt{3}}\Omega_1}}{j\Omega_1} - 2\sqrt{3}\left[\frac{\frac{2}{\sqrt{3}}e^{j\frac{2}{\sqrt{3}}\Omega_1} - \frac{1}{\sqrt{3}}e^{j\frac{1}{\sqrt{3}}\Omega_1}}{j\Omega_1} + \frac{e^{j\frac{2}{\sqrt{3}}\Omega_1} - e^{j\frac{1}{\sqrt{3}}\Omega_1}}{\Omega_1^2}\right]$$

then $h(\Omega_1, 0) = \dfrac{4\sin\Omega_1/\sqrt{3}}{\Omega_1} + 4\left[\dfrac{2\sin\frac{2}{\sqrt{3}}\Omega_1}{\Omega_1} - \dfrac{2\sin\frac{1}{\sqrt{3}}\Omega_1}{\Omega_1}\right]$

$$-2\sqrt{3}\left[\frac{\frac{2}{\sqrt{3}}\cdot 2\sin\frac{2}{\sqrt{3}}\Omega_1}{\Omega_1} - \frac{\frac{1}{\sqrt{3}}\cdot 2\sin\frac{1}{\sqrt{3}}\Omega_1}{\Omega_1} + \frac{2\cos\frac{2}{\sqrt{3}}\Omega_1 - 2\cos\frac{1}{\sqrt{3}}\Omega_1}{\Omega_1^2}\right]$$

$$= \cancel{\frac{4\sin\Omega_1/\sqrt{3}}{\Omega_1}} + \cancel{\frac{8\sin\frac{2}{\sqrt{3}}\Omega_1}{\Omega_1}} - \cancel{\frac{8\sin\frac{1}{\sqrt{3}}\Omega_1}{\Omega_1}} - \cancel{\frac{8\sin\frac{2}{\sqrt{3}}\Omega_1}{\Omega_1}} + \cancel{\frac{4\sin\frac{1}{\sqrt{3}}\Omega_1}{\Omega_1}}$$

$$- \frac{4\sqrt{3}\cos\frac{2}{\sqrt{3}}\Omega_1}{\Omega_1^2} + \frac{4\sqrt{3}\cos\frac{1}{\sqrt{3}}\Omega_1}{\Omega_1^2}$$

$$= \frac{4\sqrt{3}}{\Omega_1^2}\left[\cos\frac{2}{\sqrt{3}}\Omega_1 + \cos\frac{1}{\sqrt{3}}\Omega_1\right]$$

$$= \frac{4\sqrt{3}}{\Omega_1^2}\left[2\sin\frac{1}{2}\left(\frac{3}{\sqrt{3}}\Omega_1\right)\sin\frac{1}{2}\left(\frac{1}{\sqrt{3}}\Omega_1\right)\right]$$

$$= 8\sqrt{3}\ \frac{\sin\frac{\sqrt{3}}{2}\Omega_1}{\Omega_1}\ \frac{\sin\frac{1}{2\sqrt{3}}\Omega_1}{\Omega_1} \quad //$$

$$H(0,\Omega_2)\big|_{R_1} = \frac{4}{\sqrt{3}}\frac{\sin\Omega_2}{\Omega_2}$$

$$H(0,\Omega_2)\big|_{R_3} = \int_{1/\sqrt{3}}^{2/\sqrt{3}}\int_{-(2-\sqrt{3}t_1)}^{2-\sqrt{3}t_1} e^{-j\Omega_2 t_2}\,dt_1\,dt_2$$

$$= \int_{1/\sqrt{3}}^{2/\sqrt{3}} \frac{2\sin[(2-\sqrt{3}t_1)\Omega_2]}{\Omega_2}\,dt_1$$

$$= \frac{2}{\Omega_2}\int_{1/\sqrt{3}}^{2/\sqrt{3}}[\sin 2\Omega_1\cos\sqrt{3}t_1\Omega_2 - \sin\sqrt{3}t_1\Omega_2\cos 2\Omega_1]\,dt_1$$

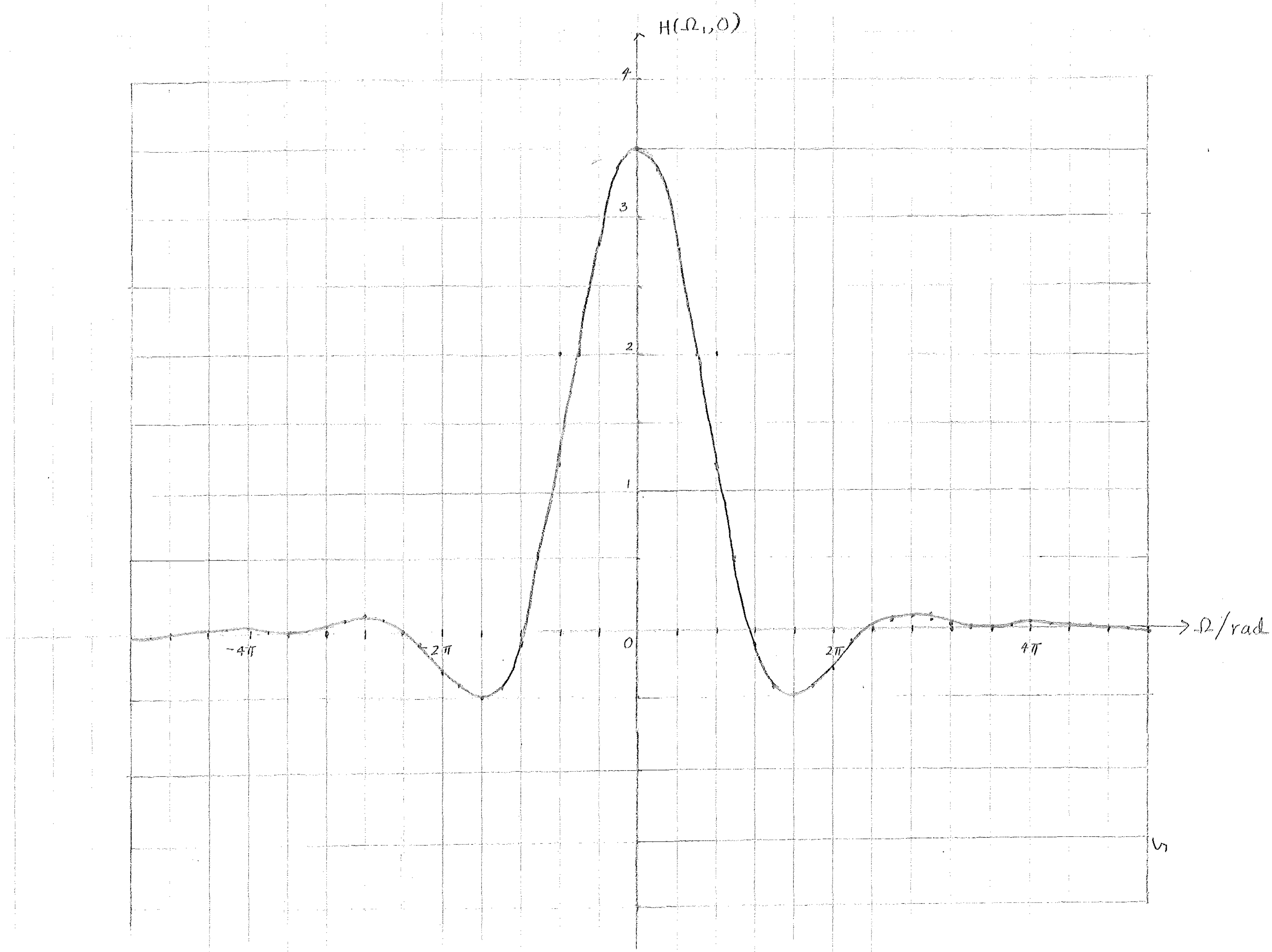
$$= \frac{2}{\Omega_2}\left[\sin 2\Omega_2\frac{\sin(2\Omega_2)-\sin(\Omega_2)}{\sqrt{3}\Omega_2} + \cos 2\Omega_2\frac{\cos(2\Omega_2)-\cos(\Omega_2)}{\sqrt{3}\Omega_2}\right]$$

$$= \frac{2}{\Omega_2}\left[\frac{1}{\sqrt{3}\Omega_2} - \frac{\cos\Omega_2}{\sqrt{3}\Omega_2}\right]$$

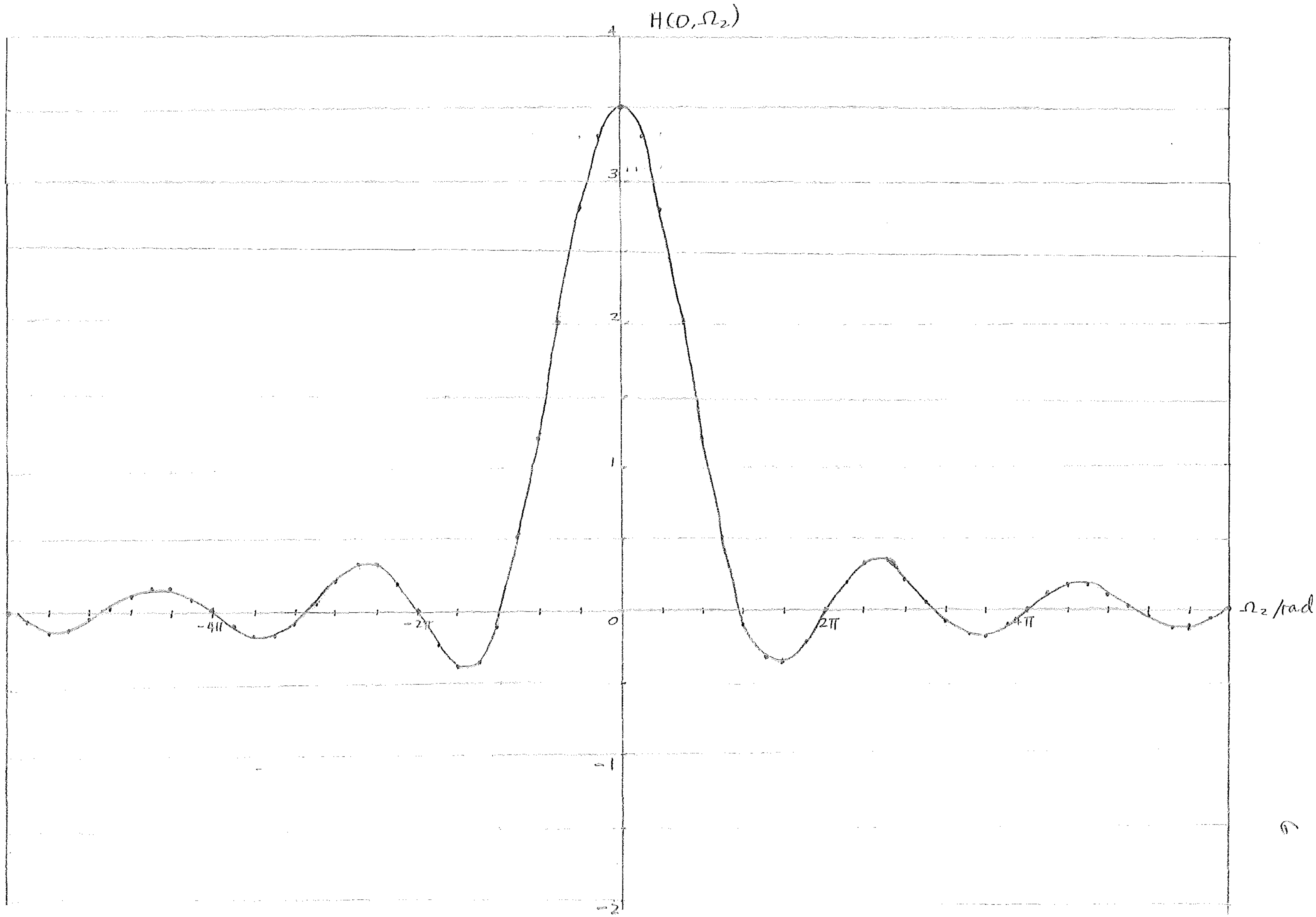
$$= \frac{2}{\sqrt{3}\Omega_2^2}(1-\cos\Omega_2) = \frac{4\sin^2\Omega_2/2}{\sqrt{3}\Omega_2^2} = \frac{1}{\sqrt{3}}\frac{\sin^2\Omega_2/2}{(\Omega_2/2)^2}$$

$$H(0,\Omega_2)\big|_{R_2} = H(0,\Omega_2)\big|_{R_1} = \frac{1}{\sqrt{3}}\frac{\sin^2\Omega_2/2}{(\Omega_2/2)^2}$$

then $H(0,\Omega_2) = \frac{4}{\sqrt{3}}\frac{\sin\Omega_2}{\Omega_2} + \frac{2}{\sqrt{3}}\frac{\sin^2\Omega_2/2}{(\Omega_2/2)^2}$ //



H(Ω₁,0)
4
3
2
1
0
-4π
-2π
2π
4π
Ω/rad
5



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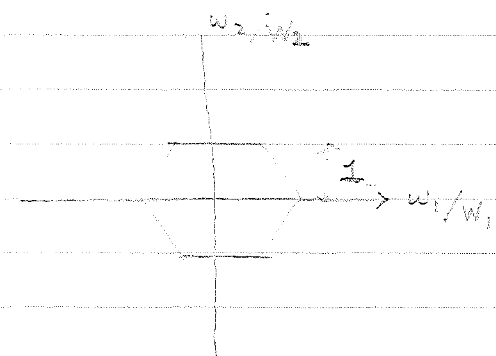
c) By piece-slice theorem:

the slice of $H\left(\frac{\Omega_2}{\Omega_1} = \frac{1}{\sqrt{3}}\right) = H(0, \Omega_2)$

$$H\left(\frac{\Omega_2}{\Omega_1} = -\sqrt{3}\right) = H(\Omega_1, 0)$$

d) Given $H(\omega_1, \omega_2) = hex\left(\frac{\omega_1}{W_1}, \frac{\omega_2}{W_2}\right)$

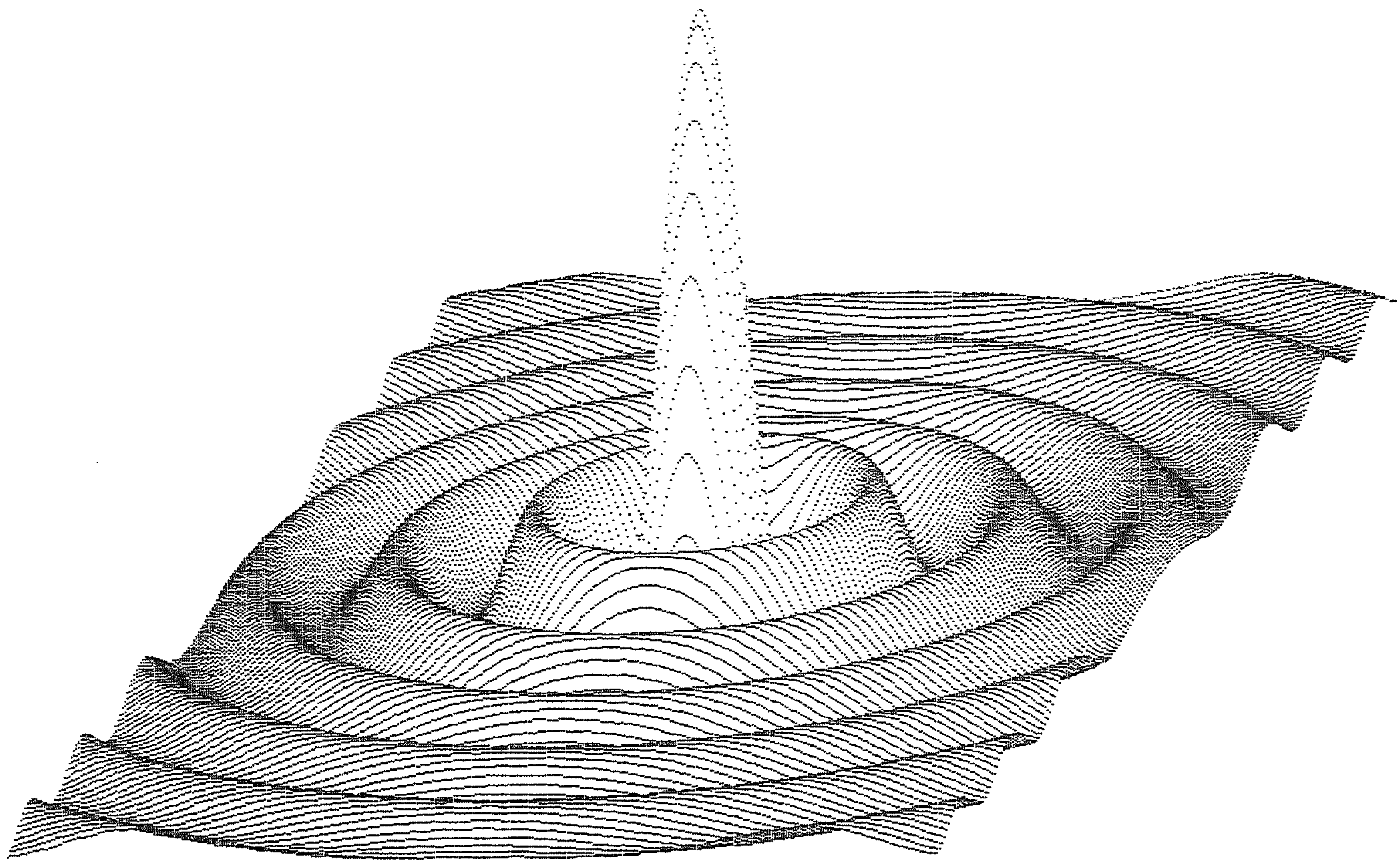


Using the result in a), we get

$$h[n_1, n_2] = \frac{W_1 W_2}{(2\pi)^2}\left[H\left(\Omega_1 = -n_1 W_1, \Omega_2 = -n_2 W_2\right)\right]$$

$$= \frac{W_1 W_2}{(2\pi)^2}\left[\frac{4\sin(W_2 n_2)\sin(W_1 n_1/\sqrt{3})}{W_1 W_2 n_1 n_2} - \frac{2\cos\left[2W_1 n_1/\sqrt{3}\right]}{W_2 n_2(\sqrt{3}W_2 n_2 + W_1 n_1)} + \frac{2\cos\left[W_2 n_2 + W_1 n_1/\sqrt{3}\right]}{W_2 n_2(\sqrt{3}W_2 n_2 + W_1 n_1)}\right.$$

$$\left. - \frac{2\cos(W_2 n_2 - W_1 n_1/\sqrt{3})}{W_2 n_2(\sqrt{3}W_2 n_2 + W_1 n_1)} + \frac{2\cos\left[2W_1 n_1/\sqrt{3}\right]}{W_2 n_2(\sqrt{3}W_2 n_2 + W_1 n_1)}\right] \quad //$$



hinc (Ω_1, Ω_2)
(from solutions of D. Arpin)

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3

Problem 2.3:

a) The easiest way to attack this problem could well be to simply try several examples.

EX: $N_1=N_2=N$, PERIOD = N

$N_1=RN_2$, PERIOD = N_1

N_1,N_2 REL PRIME , PERIOD = N_1N_2

In general, the period of $\tilde{x}(n,n)$ is $\dfrac{N_1N_2}{\gcd(N_1,N_2)}$ where $\gcd(N_1,N_2)$ is the greatest common divisor of N_1 & N_2.

b)

$$\tilde{X}_1(k) = \sum_{n=0}^{N_1N_2-1} \tilde{x}_2(n,n)W_{N_1N_2}^{nk}$$

$$= \sum_{n=0}^{N_1N_2-1} \cdot \frac{1}{N_1N_2} \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} \tilde{X}_2(k_1,k_2)W_{N_1}^{-nk_1}W_{N_2}^{-nk_2}W_{N_1N_2}^{nk}$$

$$= \frac{1}{N_1N_2} \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} \tilde{X}_2(k_1,k_2) \sum_{n=0}^{N_1N_2-1} [W_{N_1}^{-k_1}W_{N_2}^{-k_2}W_{N_1N_2}^{k}]^n$$

The innermost sum is zero unless $k=N_1k_2+N_2k_1$

$$\therefore \quad \tilde{X}_1(k) = \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} \tilde{X}_2(k_1,k_2)\delta(k-N_1k_2-N_2k_1)$$

Since N_1 & N_2 are relatively prime, each value of (k_1,k_2) over the range of summation contributes to only one value of k. The samples in $\tilde{X}_1(k)$ are simply the samples of $\tilde{X}_2(k_1,k_2)$ scrambled.

3

Problem 2.3:

a) The easiest way to attack this problem could well be to simply try several examples.

EX: $N_1=N_2=N$, PERIOD = N

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In general, the period of $\tilde{x}(n,n)$ is $\frac{N_1N_2}{\gcd(N_1,N_2)}$ where $\gcd(N_1,N_2)$ is the greatest common divisor of N_1 & N_2.

b)

$$\tilde{X}_1(k) = \sum_{n=0}^{N_1N_2-1} \tilde{x}_2(n,n)W_{N_1N_2}^{nk}$$

$$\therefore = \sum_{n=0}^{N_1N_2-1} \cdot \frac{1}{N_1N_2} \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} \tilde{X}_2(k_1,k_2)W_{N_1}^{-nk_1}W_{N_2}^{-nk_2}W_{N_1N_2}^{nk}$$

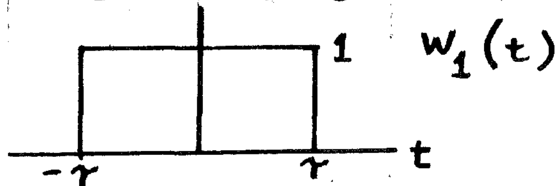
$$= \frac{1}{N_1N_2} \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} \tilde{X}_2(k_1,k_2) \sum_{n=0}^{N_1N_2-1} [W_{N_1}^{-k_1}W_{N_2}^{-k_2}W_{N_1N_2}^{k}]^n$$

The innermost sum is zero unless $k=N_1k_2+N_2k_1$

$$\therefore \tilde{X}_1(k) = \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} \tilde{X}_2(k_1,k_2)\delta(k-N_1k_2-N_2k_1)$$

Since N_1 & N_2 are relatively prime, each value of (k_1,k_2) over the range of summation contributes to only one value of k. The samples in $\tilde{X}_1(k)$ are simply the samples of $\tilde{X}_2(k_1,k_2)$ scrambled.

1. For the one dimensional window:



compute the leakage-resolution tradeoff for the following 2-D generalizations:

1. Outer Product (Along the 45° line)*.
2. Rotated Window
3. Rotated Spectrum Window

* Note, $\frac{\sin^2(\pi x)}{(\pi x)^2}$ does not have a relative maximum at $x = 3/2$.

2. The Parzen window is obtained by convolving a triangular (Bartlett) window with itself and scaling. The result is:

$$W_1(t) = \begin{cases} 1 - 6\left|\frac{t}{\tau}\right| + 6\left|\frac{t}{\tau}\right|^3 ; & |t| \le \frac{\tau}{2} \\ 2\left[1 - \left|\frac{t}{\tau}\right|\right]^3 ; & \frac{\tau}{2} \le |t| < \tau \\ 0 ; & |t| > \tau \end{cases}$$

(a) Compute the corresponding rotated spectrum window.

(b) Plot $W_1(t)$ and your result in part (a) on the same axis for $\tau = 1$.

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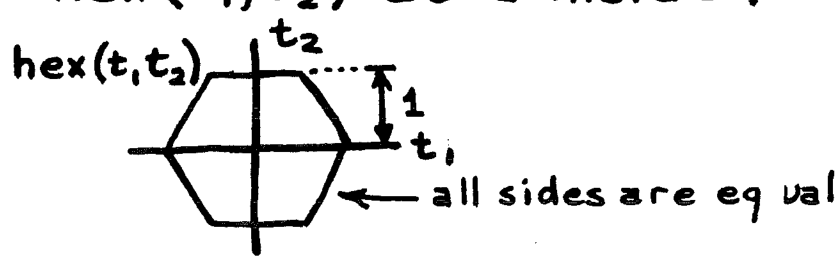
3. Define hex(t_1, t_2) as 1 inside & 0 outside:



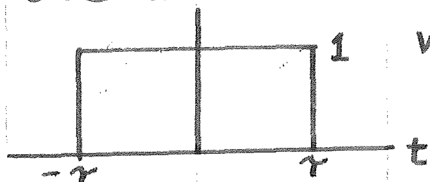
Let the 2-D Fourier transform be hinc(Ω_1, Ω_2)

(a) Compute hinc (Ω_1, Ω_2)
(b) Evaluate and sketch $h(\Omega_1, 0)$ and $h(0, \Omega_2)$
(c) What other 1-D slices of hinc (Ω_1, Ω_2) are equivalent to the slices in (b)?
(d) A 2-D filter has a frequency response

$$H(\omega_1, \omega_2) = \text{hex}\left(\frac{\omega_1}{W_1}, \frac{\omega_2}{W_2}\right)$$

Compute the corresponing impulse response, $h[n_1 n_2]$

1. For the one dimensional window:



compute the leakage-resolution tradeoff for the following 2-D generalizations:

1. Outer Product (Along the 45° line)*.
2. Rotated Window
3. Rotated Spectrum Window

* Note, $\frac{\sin^2(\pi X)}{(\pi X)^2}$ does not have a relative maximum at $X = 3/2$.

2. The Parzen window is obtained by convolving a triangular (Bartlett) window with itself and scaling. The result is:

$$w_1(t) = \begin{cases} 1 - 6\left|\frac{t}{\tau}\right|^2 + 6\left|\frac{t}{\tau}\right|^3 ; & |t| \le \frac{\tau}{2} \\ 2\left[1 - \left|\frac{t}{\tau}\right|\right]^3 ; & \frac{\tau}{2} \le |t| < \tau \\ 0 ; & |t| > \tau \end{cases}$$

(a) Compute the corresponding rotated spectrum window.

(b) Plot $w_1(t)$ and your result in part (a) on the same axis for $\tau = 1$.

Final Examination: EE521

Robert J. Marks II

- Do all of your work in this test booklet.
- The test begins promptly at 8:30 AM.
- The test is closed book and closed notes. Each student is allowed two $8\frac{1}{2} \times 11$ sheet of paper with notes. Calculators are allowed.
- Each problem is worth the same number of points.
- After the test, you may forget about this course for the rest of the year.

1. The first problem is your work on the McClellan transform. Please attach it to this booklet when you hand in your test.

2. Provide a detailed sketch of the projection of

$$x(t_1, t_2) = \Pi\left(\frac{t_1}{2}\right)\left(\frac{t_2}{2}\right)$$

(a) onto the t_2 axis,

(b) perpendicular to the line $t_1 = t_2$,

3. Denote an Abel transform, $f_A(t)$, of a radial function, $f(r)$, by

$$f(r) \leftrightarrow f_A(t).$$

(a) What is the scaling theorem for Abel transforms? In other words,

$$f\left(\frac{r}{M}\right) \leftrightarrow ?$$

You may assume that $M > 0$.

(b) Given the Abel transform pair

$$\Pi(r) \leftrightarrow \left(1 - 4t^2\right)^{\frac{1}{2}} \Pi(t),$$

evaluate the Abel transform of the annulus

$$f(r) = \begin{cases} 1 & ; 1 \le r \le 2 \\ 0 & ; \text{otherwise} \end{cases}$$

4. Consider the component filter (transformation function)

$$F(\omega_1, \omega_2) = \cos\left(\frac{\omega_1 - \omega_2}{2}\right).$$

In the $2\pi \times 2\pi$ square in the (ω_1, ω_2) plane, we desire a two dimensional filter

$$H(\omega_1, \omega_2) = \begin{cases} 1 & ; |\omega_1 - \omega_2| \leq \frac{\pi}{2} \\ 0 & ; \text{otherwise} \end{cases}$$

Make a **detailed sketch** of the prototype filter

$$H(\omega) = \sum_{n=0}^{N} a_n \cos(n\omega).$$

(Do **not** evaluate any values for the a_n's.)

5. The IIR filter $H(\omega_1,\omega_2)$ is iteratively implemented where

$$B(\omega_1,\omega_2) = \frac{1}{H(\omega_1,\omega_2)} = 1 - \frac{1}{2}\cos^2(\omega_1)\cos^2(\omega_2).$$

Evaluate the required number of iterations, I, required to assure the maximum error of both the output and the corresponding transfer function does not exceed $\frac{1}{256}$.

1. Scratch Paper

2. Scratch Paper

3. Scratch Paper

DEPARTMENT OF ELECTRICAL ENGINEERING
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EE595

Take Home Final

Solutions

1. Using the McClellan transform, design a 2-D hexagonal FIR low pass filter with near circular symmetry that passes frequencies $|\omega| \le \pi/4$. Plot the frequency response slices $H(\omega_1, 0)$ and $H(0, \omega_2)$.

2. Page 280, #5.3.

3. An $M > 1$ dimensional signal has a spectrum with the support of a hypersphere with radius ρ. The signal is sampled at minimum density and a sample is lost at the origin. The known data is perturbed by zero mean stationary sample wise white noise with variance $\overline{\xi^2}$. Plot the restoration noise level, $\overline{n^2(\vec{0})} / \overline{\xi^2}$ for $1 < M \le 8$.

4. Page 342, #6.8.

From the work of T. Ku

1.)

$$\cos\omega = F_H(\omega_1, \omega_2)$$

$$= A + B\cos\frac{2\omega_1}{\sqrt{3}} + C\cos\left(\frac{\omega_1}{\sqrt{3}} + \omega_2\right) + D\cos\left(\frac{\omega_1}{\sqrt{3}} - \omega_2\right)$$

$$A = -\frac{1}{3} \qquad B = C = D = \frac{4}{9}$$

Choose $H(\omega) = 1 \quad |\omega| \le \frac{\pi}{4}$

$= 0$ otherwise

$$h(n) = \frac{1}{2\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left(\frac{e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n}}{jn}\right) = \frac{\sin\frac{\pi}{4}n}{\pi n}$$

Let $N = 100$

$$H(\omega_1, \omega_2) = \sum_{n=0}^{N} a(n)\, T_n[F(\omega_1, \omega_2)]$$

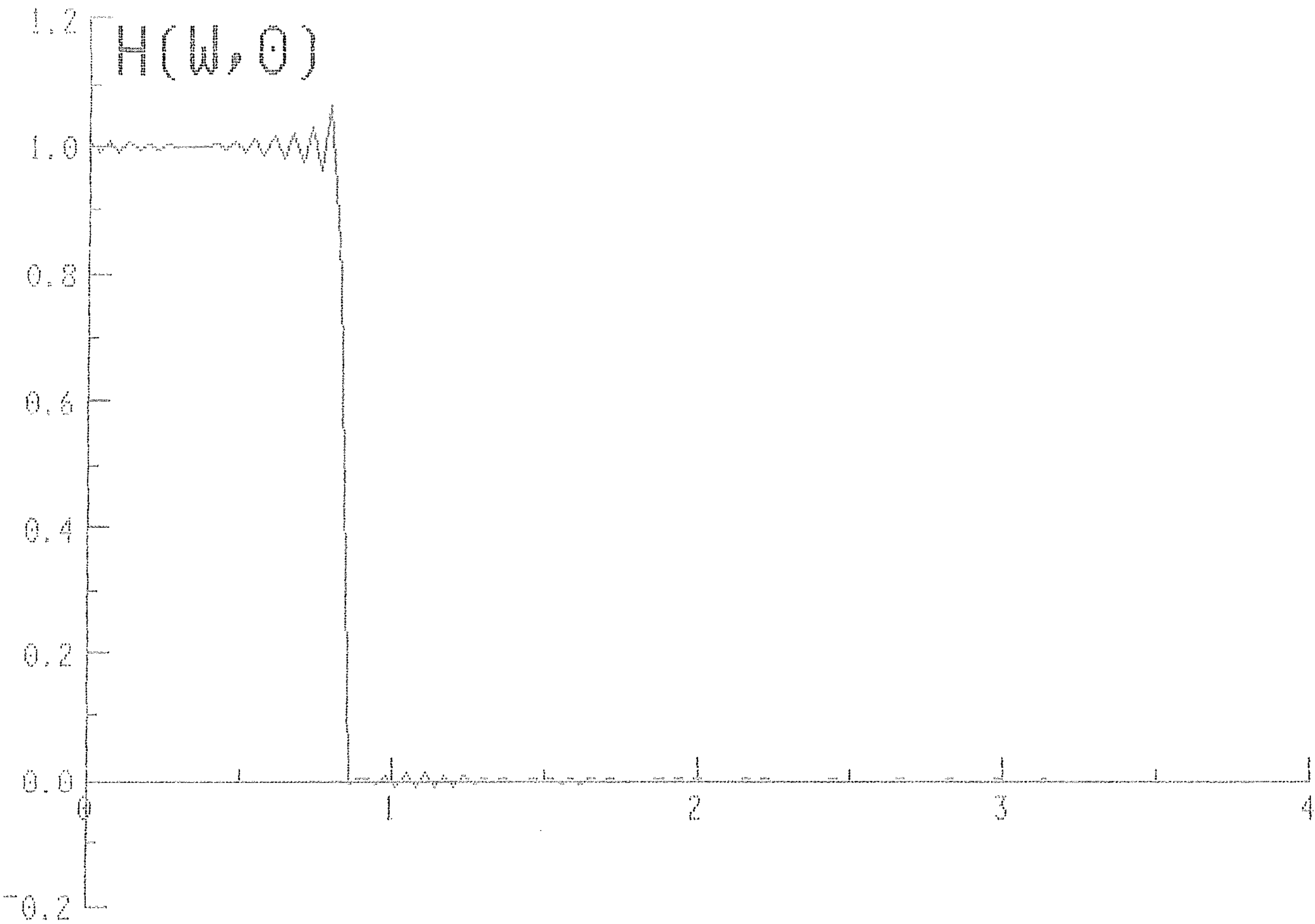
where $a(n) = \begin{cases} h(0) & n = 0 \\ 2h(n) & n > 0. \end{cases}$

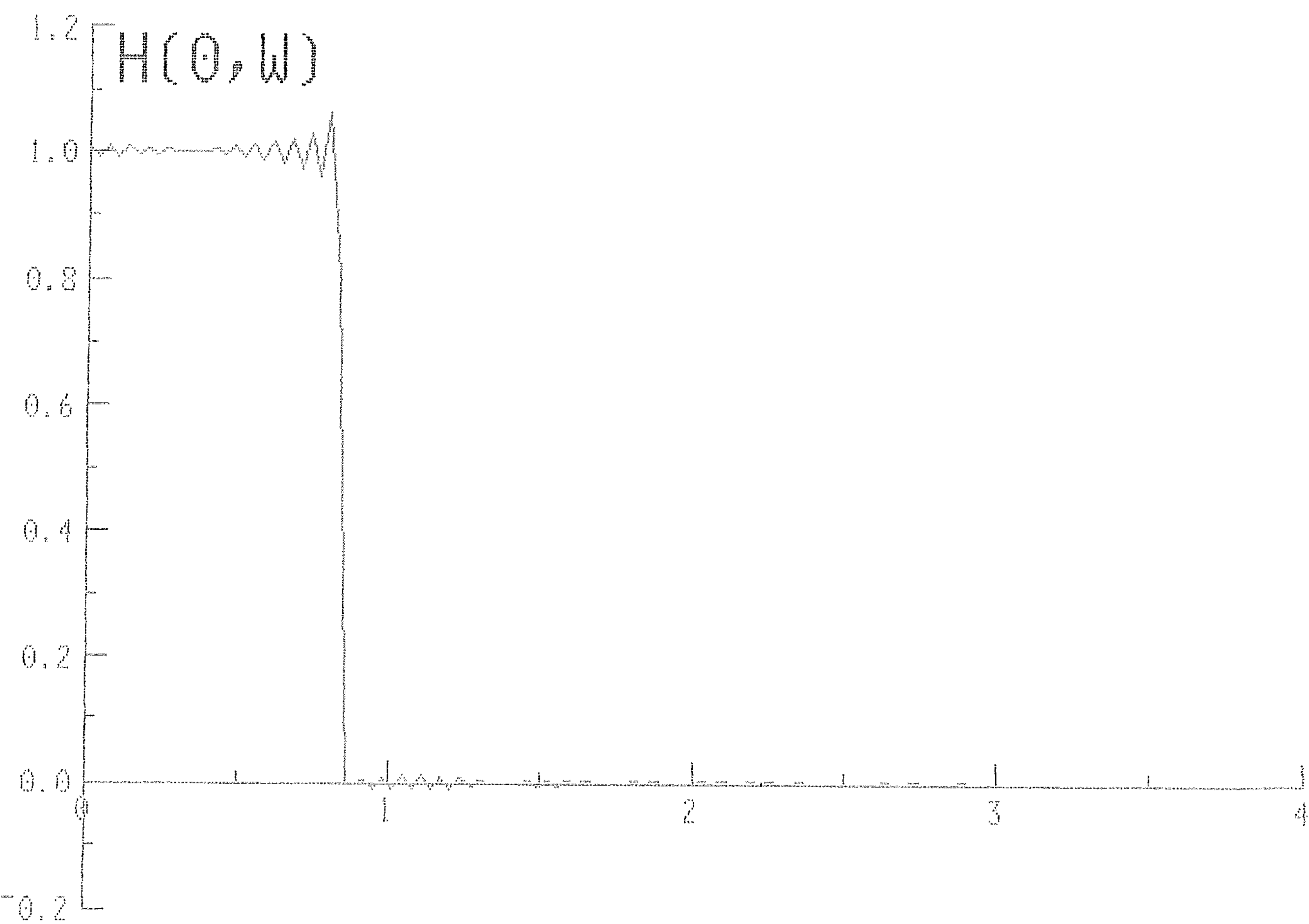
$T_0[X] = 1$ and $T_1[X] = X$. $T_n[X] = 2X\, T_{n-1}[X] - T_{n-2}[X]$.

$$H(\omega_1, 0) = \sum_{n=0}^{N} a(n)\, T_n\left[-\frac{1}{3} + \frac{4}{9}\cos\frac{2\omega_1}{\sqrt{3}} + \frac{8}{9}\cos\left(\frac{\omega_1}{\sqrt{3}}\right)\right]$$

$$H(0, \omega_2) = \sum_{n=0}^{N} a(n)\, T_n\left[-\frac{1}{3} + \frac{8}{9}\cos\left(\frac{\omega_1}{\sqrt{3}}\right)\right]$$

$H(\omega_1, 0)$ and $H(0, \omega_2)$ plot as follow.


H(W,0)
1.2
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
0
1
2
3
4


H(0,W)
1.2
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
0
1
2
3
4

1 Using the McClellan transform, design a 2-D hexagonal FIR low pass filter with 'near circular' symmetry.

$$H_{ideal}(\omega_1, \omega_2) = \begin{cases} 1 & |\omega| < \pi/4 \\ 0 & \text{otherwise} \end{cases}$$

Note: for the hexagonal filter case, the ω space has been redefined to make the hexagon be "non stretched"

$$X(\omega_1,\omega_2) = \sum_{n_1}\sum_{n_2} x[n_1, n_2] \exp j\left(\frac{2n_1 - n_2}{\sqrt{3}}\omega_1 + n_2\omega_2\right)$$

10th order (From work of W.H. Nicholls)

The transform is accomplished

A find the 1-D prototype filter
(its order was not specified, I chose order = 10)

B find the 2-D transformation function $F(\omega_1, \omega_2)$

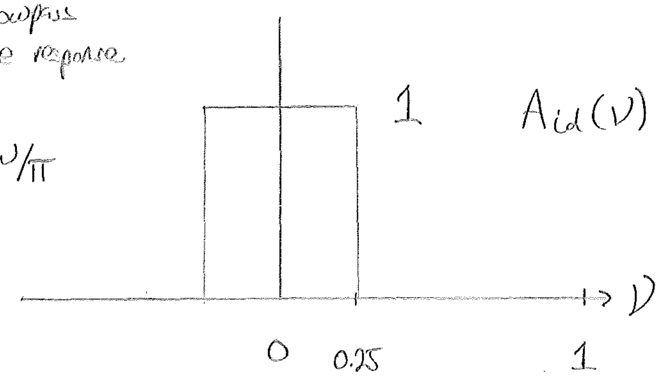
C see if it worked

A

the coefficients ($h[n]$) of the prototype low pass filter may be found using the fourier transform method (Stanley section 8-2)

ideal lowpass amplitude response

$\nu = \omega/\pi$



The coefficients may be found from the integral:

$$h[m] = \int_0^{0.25} \cos m\pi\nu \, d\nu = \frac{\sin 0.25 m\pi}{m\pi} \quad \text{(Stanley 225)}$$

the coefficients of a third order filter are *

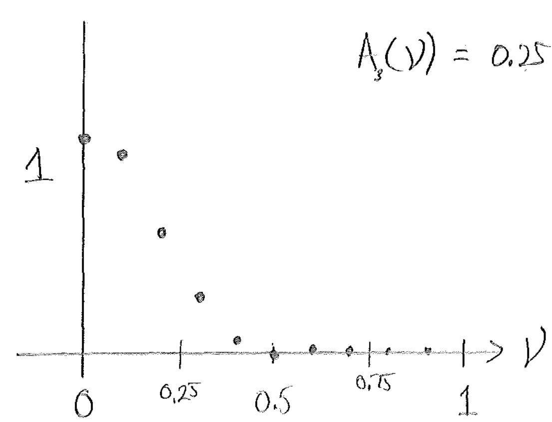
$$h[0] = 0.250 \qquad h[\pm 1] = 0.225 \qquad h[\pm 2] = 0.159 \qquad h[\pm 3] = 0.075$$

$$2h[1] = 0.450 \qquad 2h[2] = 0.318 \qquad 2h[3] = 0.150$$

third order amplitude response is:

$$A_3(\nu) = 0.25 + \sum_{n=1}^{3} 2h[n] \cos n\pi\nu$$

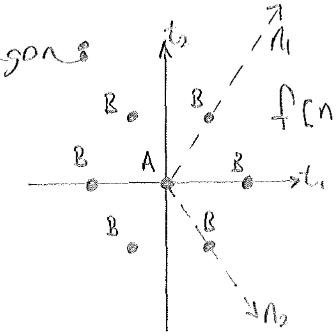
ν	$A(\nu)$
0.0	1.168
0.1	1.023
0.2	0.666
0.3	0.273
0.4	0.010
0.5	-0.068
0.6	
0.7	
0.8	



* see computer printout for 10th order filter

B find the 2-D transformation function $F(\omega_1,\omega_2) = \cos\omega$

the simplest choice of a hexagonal, circular F is the frequency response of a weighted delta plus a weighted unit hexagon:



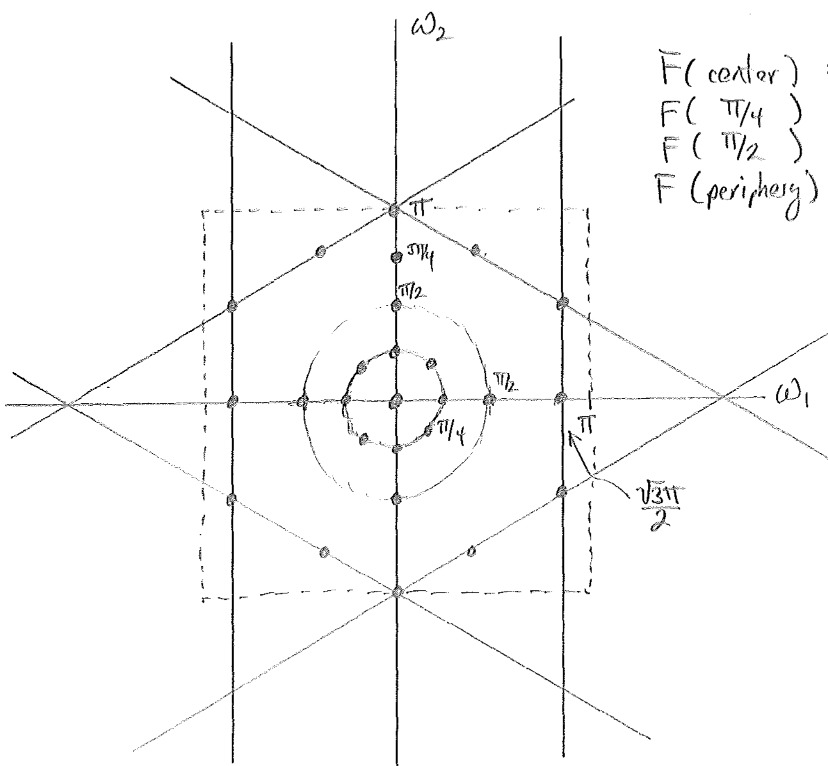
f's frequency response is:

$$
\begin{aligned}
F(\omega_1,\omega_2) = \; & f[0,0] \\
& + f[1,0]\exp\left(-j\left(2\omega_1/\sqrt{3}\right)\right) + f[-1,0]\exp\left(-j\left(-2\omega_1/\sqrt{3}\right)\right) \\
& + f[0,1]\exp\left(-j\left(-\omega_1/\sqrt{3}+\omega_2\right)\right) + f[0,-1]\exp\left(-j\left(\omega_1/\sqrt{3}-\omega_2\right)\right) \\
& + f[1,1]\exp\left(-j\left(\omega_1/\sqrt{3}+\omega_2\right)\right) + f[-1,-1]\exp\left(-j\left(-\omega_1/\sqrt{3}-\omega_2\right)\right)
\end{aligned}
$$

which simplifies to:

$$F(\omega_1,\omega_2) = A + 2B\cos 2\omega_1/\sqrt{3} + 4B\cos\omega_2\cos\omega_1/\sqrt{3}$$

an isopotential plot of this response:



$F(\text{center}) = A + 6B$
$F(\pi/4) = A + B \cdot 4.828$
$F(\pi/2) = A + 2B$
$F(\text{periphery}) = A - 2B$

B cont'd

To select values for A and B note that since

$$F(\omega_1,\omega_2) = \cos\omega,$$ and our prototype is lowpass,

F should approach 1 in the passband and -1 in the stopband.

The most natural selection of A and B therefore is

$$A = -2B$$
$$B = 1/4$$

This choice produces

$$F(\text{center, passband}) = 1$$
$$F(\text{periphery, stopband}) = -1$$

$$F(\omega_1,\omega_2) = -1/2 + 1/2\cos(2\omega_1/\sqrt{3}) + \cos\omega_2\cos(\omega_1/\sqrt{3})$$

note

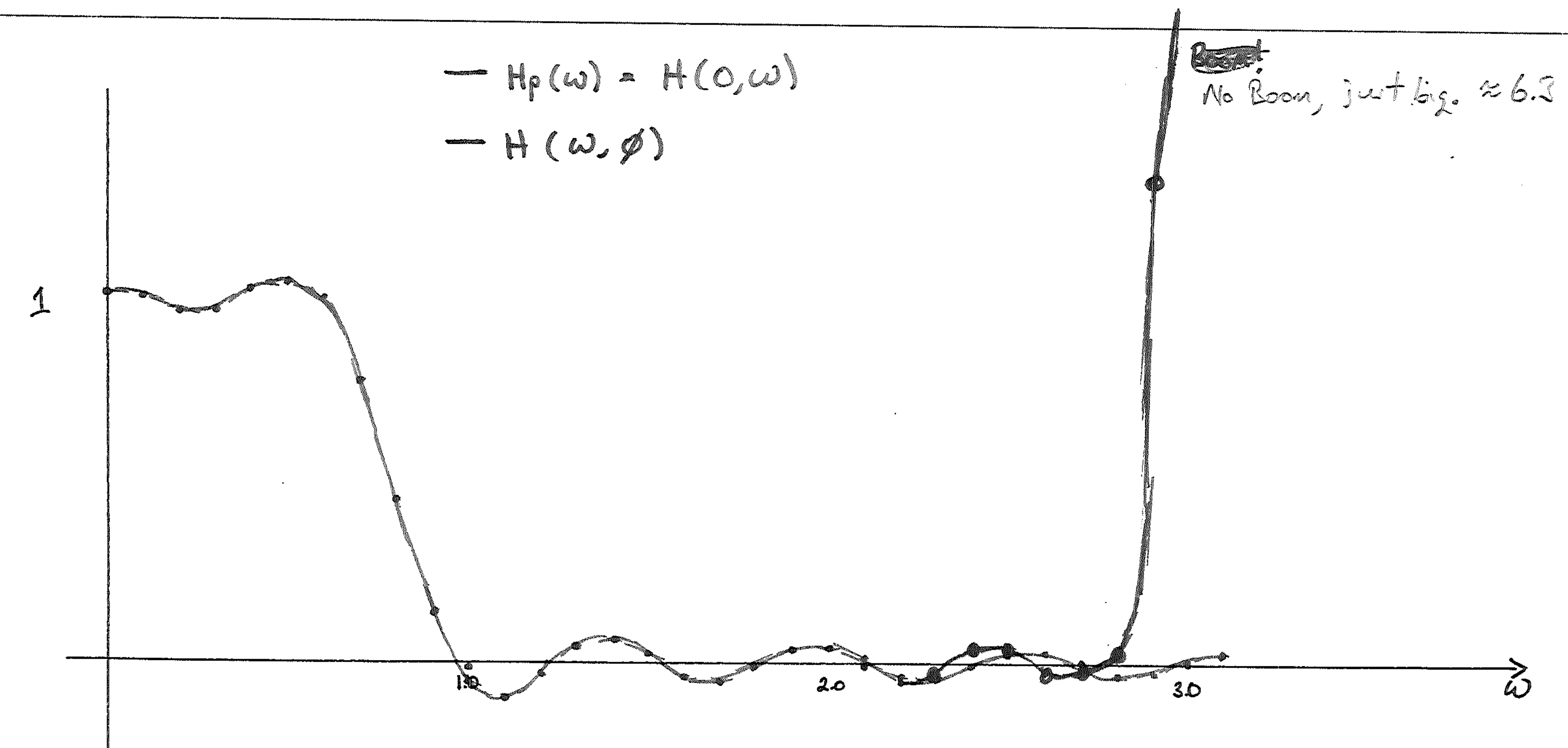
$$F(0,\omega_2) = \cos\omega_2$$ the ω_2 slice exactly corresponds

$$\Rightarrow H(0,\omega_2) = H(\omega_2)$$ to the prototype H

but

$$F(\omega_1,\emptyset) = -1/2 + 1/2\cos(2\omega_1/\sqrt{3}) + \cos(\omega_1/\sqrt{3})$$

so, $H(\omega_1,\emptyset)$ will be different from the prototype c did the transform work?



notes

$H_p(\omega) = H(0, \omega)$ for all ω $-\pi < \omega < \pi$

$H_p(\omega) \neq H(\omega, 0)$ after ω exceeds 2.0 the functions diverge and H becomes relatively large perhaps modifying A and B could help some

See printout sheets. Set $A = -0.375$ $B = 0.229$ removes the "Boom" but does very bad things to the low frequency match of Hp and H

Problem 6.8:

a) $$W'(\underline{k}) = \frac{1}{N}\sum_{i=0}^{N-1} w(i)\,\exp\,[-j\underline{k}'(\underline{x}_i+\underline{d})]$$

$$= \exp[-j\underline{k}'\underline{d}]\sum_{i=0}^{N-1} w(i)\,\exp[-j\underline{k}'\underline{x}_i]$$

$$= W(\underline{k})\,\exp[-j\underline{k}'\underline{d}]$$

where $\underline{d} \triangleq (d_x, d_y, d_z)'$

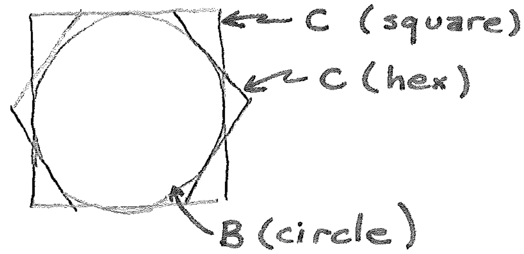
b) $$W'(\underline{k}) = \frac{1}{N}\sum_{i=0}^{N-1} w(i)\,\exp[-jD\underline{k}'\cdot\underline{xi}]$$

$$= W(\underline{k}D)$$

c) $$W'(\underline{k}) = \frac{1}{N}\sum_{i=0}^{N-1} w(i)\,\exp[-jk_xD_xx_i - jk_yD_yy_i - jk_zD_zz_i]$$

$$= W(\underline{\ell}) \text{ where } \underline{\ell} = (k_xD_x, k_yD_y, k_zD_z)'$$

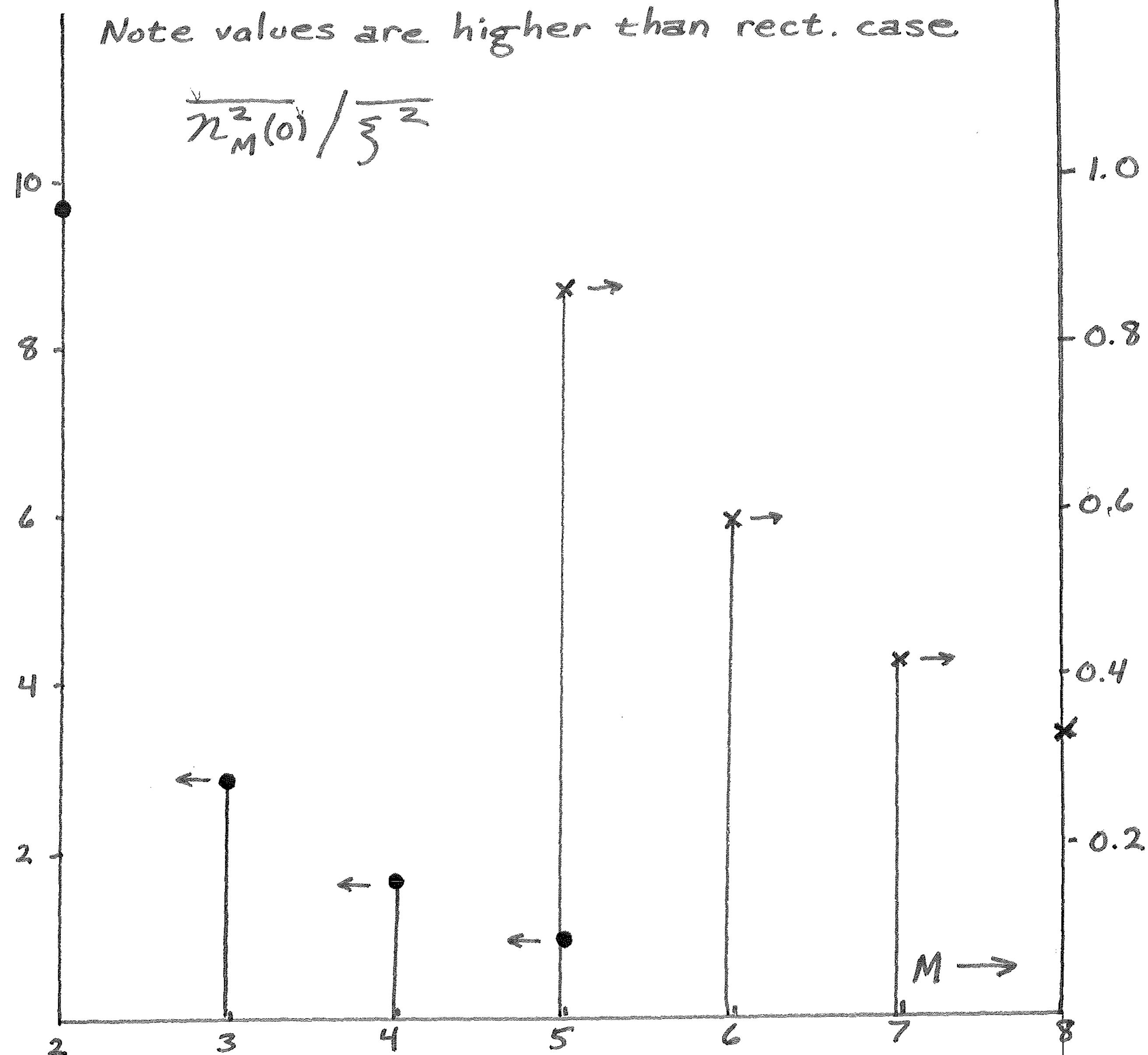
3. For (minimum density) hex sampling



$$\frac{\overline{n^2(\vec{0})}}{\overline{\xi^2}} = \left[\frac{C}{B} - 1\right]^{-1} = \left[\frac{C}{\hat{C}}\,\frac{\hat{C}}{B} - 1\right]^{-1}$$

A table of $C/\hat{C}$ is on p. 47 of text
$\hat{C} = (2\rho)^M$
$B =$ AREA OF M-D sphere

Note values are higher than rect. case



Problem 5.3:

a) $\dfrac{Y_z(z)}{X_z(z)} = \dfrac{1}{1-az^{-1}}$ $\qquad Y_z(z)=X_z(z)+az^{-1}Y_z(z)$

$$\boxed{c(n) = a\delta(n-1)}$$

b) $\boxed{y_i(n)=x(n)+ay_{i-1}(n-1)}$

c) $y_0(n)=\delta(n)$

$y_1(n)=\delta(n)+a\delta(n-1)$

$y_2(n)=\delta(n)+a\delta(n-1)+a^2\delta(n-2)$

$\vdots$

$$y_I(n) = \sum_{i=0}^{I} a^i\delta(n-i)$$

$\vdots$

$$y_\infty(n) = a^n u(n)$$

$$\boxed{e_2(n) = \sum_{n=I+1}^{\infty} a^{2n} = \frac{a^{2(I+1)}}{1-a^2}}$$

EE521

26 *Two-dimensional impulse.* State the nature of the following impulse symbols in two dimensions by giving (*a*) the locus where the impulse is located and (*b*) the linear density at each point of the locus: $\delta(x+y)$, $\delta(xy)$, $\delta(\sin\theta)$, $\delta(x^2+y^2-1)$, $\delta(x^2+y^2)$.

27 *Derivative theorems for Hankel transform.* Show that

$$(rf)' \supset -(qF)'$$

and that $f' \supset -[q\mathcal{H}\{r^{-1}f\}]'$.

28 *Derivative theorem for Hankel transform.* Show that

$$rf'(r) \supset -q^{-1}\frac{d}{dq}[q^2F(q)].$$

29 *Hankel transform theorem.* Show that

$$f(r) = \mathcal{H}\left\{q^{-1}\frac{d}{dq}\mathcal{H}\left\{r^{-1}\frac{d}{dr}f(r)\right\}\right\}.$$

30 *Hankel transform example.* Establish that the Hankel transform of $r^2\exp(-\pi r^2)$ is $(\pi^{-1}-q^2)\exp(-\pi q^2)$.

31 *Hankel transform.* Show that

$$\int_0^\infty J_1(x)J_0(ax)\,dx = H(1-a^2).$$

32 *Hankel transform example.* Verify that $(4\pi r^2)^{-1}J_2(\pi r)$ has Hankel transform $(\frac{1}{4}-q^2)\Pi(q)$.

33 *Cauchy principal value.* We often use the phrase "area under the curve $f(x)$" to mean the integral from $-\infty$ to ∞. Intuitively, from experience with areas, one might expect that the area under $f(x)$ is the same as the area under $f(x+1)$. Can you prove that

$$\int_{-\infty}^{\infty}\operatorname{sgn}x\,dx = \int_{-\infty}^{\infty}\operatorname{sgn}(x+1)\,dx?$$

34 *Radial sampling under circular symmetry.* The light from a star is received at two points spaced a certain distance q apart and the complex correlation between the two optical waveforms is determined. It can be shown that this complex number is a value of the Hankel transform $B(q)$ of the brightness distribution $b(r)$ over the stellar disk (assuming that the brightness distribution has circular symmetry). (If r is measured in radians, q will be measured in wavelengths.) Since the star is of finite extent, it suffices to sample the transform at regularly spaced distances. Show how to determine $b(r)$ from values of $B(q)$ determined at $q = 0, a, 2a, \ldots$.

35 *Abel transform.* Let $f(\cdot)$ be subjected to two Abel transformations in succession. Show that the resulting function $f_{AA}(x)$ is equal to the volume under

$f(\cdot)$ outside radius x, that is, $f_{AA}(x) = 2\pi\int_x^\infty rf(r)\,dr$. (This problem was supplied by S. J. Wernecke.)

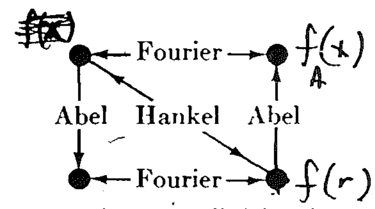
36 *Two-dimensional autocorrelation.* Let $f(r)$ have Abel transform $f_A(x)$. If we take the two-dimensional autocorrelation of $f(r)$, we get another circularly symmetrical function. Show that the Abel transform of the two-dimensional autocorrelation is the one-dimensional autocorrelation of the Abel transform $f_A(x)$; that is, $f(r) \star\star f(r)$ has Abel transform $f_A(x) \star f_A(x)$.

37 *Abel-Fourier-Hankel cycle of transforms.* Functions can be spatially arranged in groups of four to exhibit the Abel-Fourier-Hankel cycle of transforms (R. N. Bracewell, *Austral. J. Phys.*, vol. 9, p. 198, 1956, and Problem 12.16). Thus the relationships

jinc r	has Abel transform sinc x
sinc x	has Fourier transform $\Pi(q)$
$\Pi(q)$	has Hankel transform jinc r
$\Pi(q)$	has Abel transform $(1-4u^2)^{\frac{1}{2}}\Pi(u)$
$(1-2u^2)^{\frac{1}{2}}\Pi(u)$	has Fourier transform jinc r,

where jinc $r = (2r)^{-1}J_1(\pi r)$, are all compactly summarized by grouping the four functions as in the box.

$$\boxed{\begin{array}{ll} \text{jinc } r & (1-4u^2)^{\frac{1}{2}}\Pi(u) \\ \text{sinc } x & \Pi(q) \end{array}}$$



The diagram on the right is the key to the transforms implied by the spatial relationship. Verify the following important groups.

$$\boxed{\begin{array}{ll} \text{sinc } r & \Pi(u) \\ J_0(\pi x) & \pi^{-1}(\frac{1}{4}-q^2)^{-\frac{1}{2}}\Pi(q) \end{array}} \qquad \boxed{\begin{array}{ll} \delta(r-a) & 2\cos 2\pi au \\ 2a(a^2-x^2)^{-\frac{1}{2}}\Pi\left(\dfrac{x}{2a}\right) & 2\pi aJ_0(2\pi aq) \end{array}}$$

$$\boxed{\begin{array}{ll} M(r) & (1-u^2)^{\frac{1}{2}} - u^2\cosh^{-1}u^{-1} \\ \text{sinc}^2\, x & \Lambda(q) \end{array}} \qquad \boxed{\begin{array}{ll} e^{-\pi r^2} & e^{-\pi u^2} \\ e^{-\pi x^2} & e^{-\pi q^2} \end{array}}$$

38 Verify the composite similarity theorem for the Fourier-Abel-Hankel cycle of transforms, for $a > 0$:

If $\boxed{\begin{array}{ll} f(r) & F(u) \\ g(x) & G(q) \end{array}}$ then $\boxed{\begin{array}{ll} af(ar) & F\left(\dfrac{u}{a}\right) \\ g(ax) & a^{-1}G\left(\dfrac{q}{a}\right) \end{array}}$

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Take Home Final

Solutions

1. Using the McClellan transform, design a 2-D hexagonal FIR low pass filter with near circular symmetry that passes frequencies $|\omega| \leq \pi/4$. Plot the frequency response slices $H(\omega_1, 0)$ and $H(0, \omega_2)$.

2. Page 280, #5.3.

3. An $M > 1$ dimensional signal has a spectrum with the support of a hypersphere with radius ρ. The signal is sampled at minimum density and a sample is lost at the origin. The known data is perturbed by zero mean stationary sample wise white noise with variance $\overline{\xi^2}$. Plot the restoration noise level, $\overline{n^2(\vec{0})} / \overline{\xi^2}$ for $1 < M \leq 8$.

4. Page 342, #6.8.

From the work of T. Ku

1.) $\cos\omega = F_H(\omega_1, \omega_2)$

$$= A + B\cos\frac{2\omega_1}{\sqrt{3}} + C\cos\left(\frac{\omega_1}{\sqrt{3}} + \omega_2\right) + D\cos\left(\frac{\omega_1}{\sqrt{3}} - \omega_2\right)$$

$$A = -\frac{1}{3} \qquad B = C = D = \frac{4}{9}$$

Choose $H(\omega) = 1 \quad |\omega| \le \frac{\pi}{4}$

$= 0$ otherwise

$$h(n) = \frac{1}{2\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left(\frac{e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n}}{jn}\right) = \frac{\sin\frac{\pi}{4}n}{\pi n}$$

Let $N = 100$

$$H(\omega_1, \omega_2) = \sum_{n=0}^{N} a(n)\, T_n\left[F(\omega_1, \omega_2)\right]$$

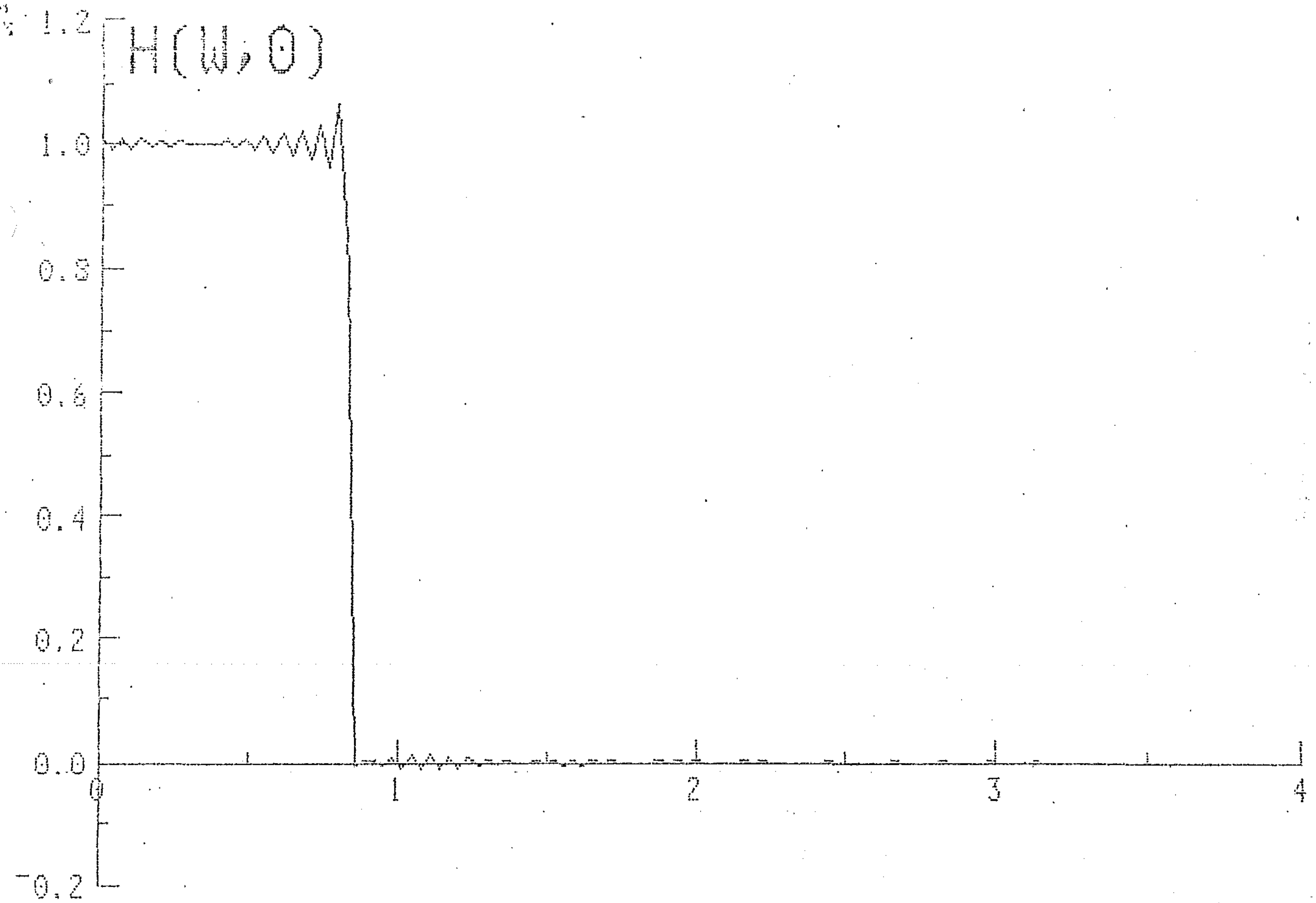
where $a(n) = \begin{cases} h(0) & n=0 \\ 2h(n) & n>0. \end{cases}$

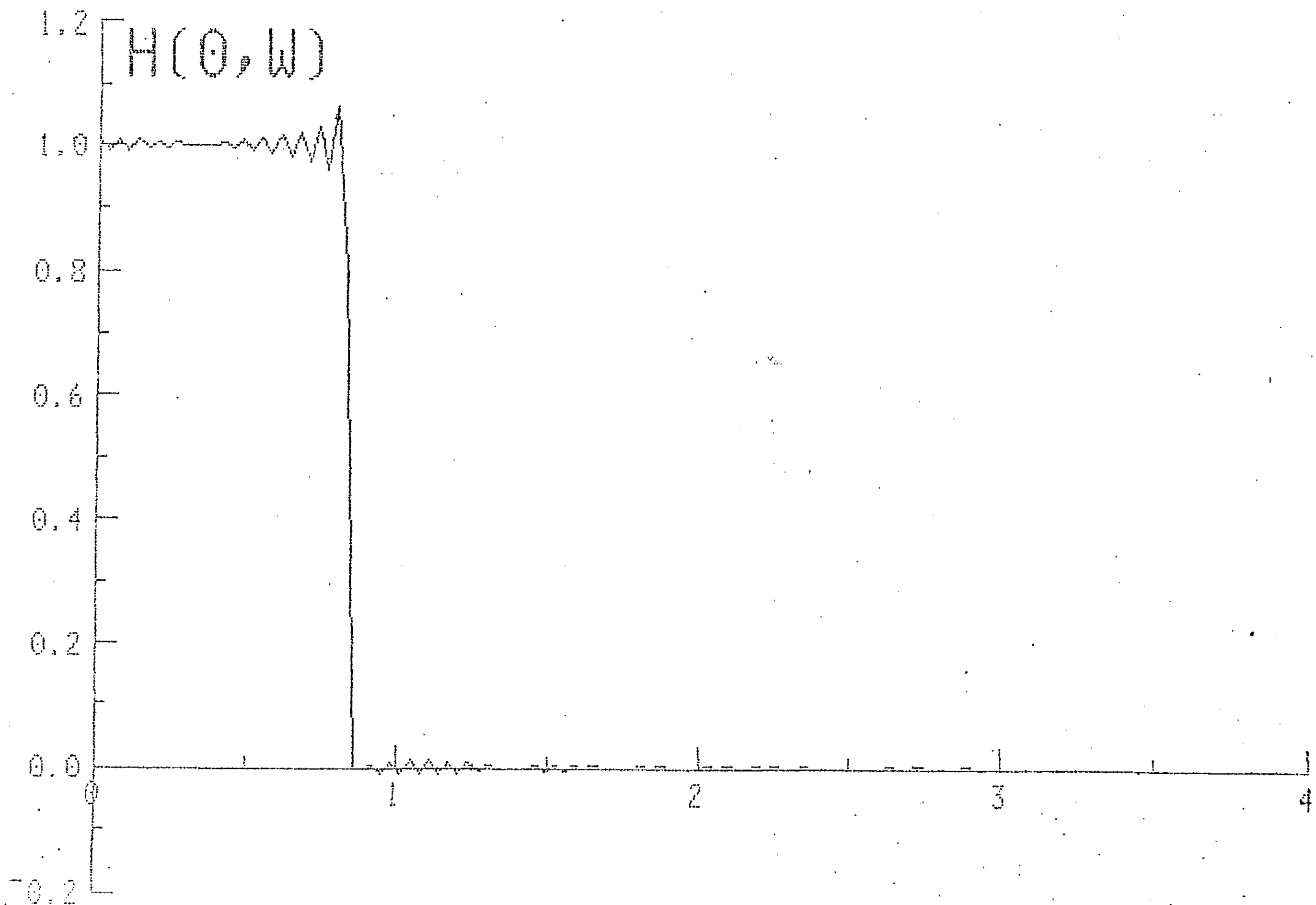
$T_0[X] = 1$ and $T_1[X] = X$. $T_n[X] = 2X\,T_{n-1}[X] - T_{n-2}[X]$

$$H(\omega_1, 0) = \sum_{n=0}^{N} a(n)\, T_n\left[-\frac{1}{3} + \frac{4}{9}\cos\frac{2\omega_1}{\sqrt{3}} + \frac{8}{9}\cos\left(\frac{\omega_1}{\sqrt{3}}\right)\right]$$

$$H(0, \omega_2) = \sum_{n=0}^{N} a(n)\, T_n\left[-\frac{1}{3} + \frac{8}{9}\cos\left(\frac{\omega_1}{\sqrt{3}}\right)\right]$$

$H(\omega_1, 0)$ and $H(0, \omega_2)$ plot as follow.


H(W,0)
1.2
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
0
1
2
3
4


H(0,W)
1.2
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
0
1
2
3
4

1. Using the McClellan transform, design a 2-D hexagonal FIR low pass filter with 'near circular' symmetry.

$$H_{ideal}(\omega_1,\omega_2) = \begin{cases} 1 & |\omega| < \pi/4 \\ 0 & \text{otherwise} \end{cases}$$

Note: for the hexagonal filter case, the ω space has been redefined to make the hexagon be "non stretched"

$$X(\omega_1,\omega_2) = \sum_{n_1}\sum_{n_2} x[n_1,n_2] \exp j\left(\frac{2n_1-n_2}{\sqrt{3}}\omega_1 + n_2\omega_2\right)$$

10th order (From work of W.H. Nicholls)

The transform is accomplished

A find the 1-D prototype filter
(its order was not specified, I chose order = 10)

B find the 2-D transformation function $F(\omega_1,\omega_2)$

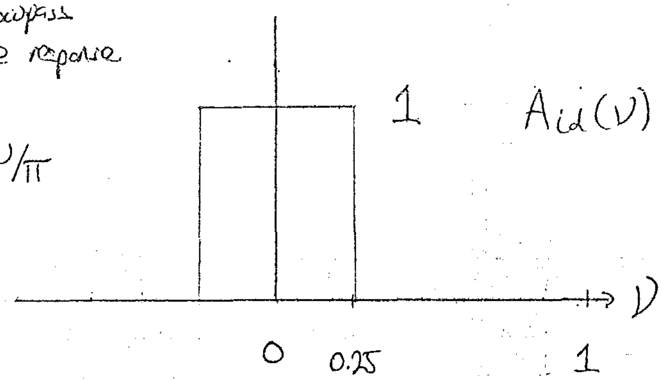
C see if it worked

A

the coefficients ($h[n]$) of the prototype low pass filter may be found using the fourier transform method (Stanley section 8-2)

ideal lowpass amplitude response

$\nu = \omega / \pi$



The coefficients may be found from the integral:

$$h[m] = \int_0^{0.25} \cos m\pi\nu \, d\nu = \frac{\sin 0.25 m\pi}{m\pi} \quad \text{(Stanley 225)}$$

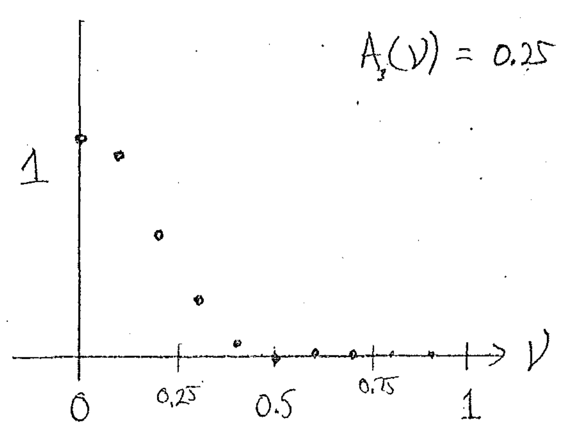
the coefficients of a third order filter are *

$$h[0] = 0.250$$
$$h[\pm 1] = 0.225 \qquad 2h[1] = 0.450$$
$$h[\pm 2] = 0.159 \qquad 2h[2] = 0.318$$
$$h[\pm 3] = 0.075 \qquad 2h[3] = 0.150$$

third order amplitude response is:

$$A_3(\nu) = 0.25 + \sum_{n=1}^{3} 2h[n] \cos n\pi\nu$$

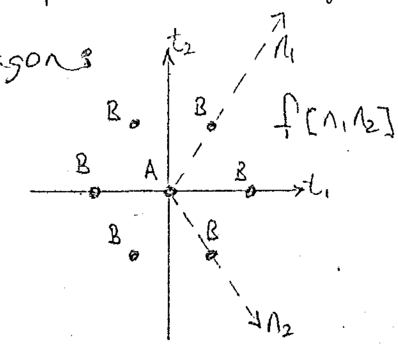
ν	$A(\nu)$
0.0	1.168
0.1	1.023
0.2	0.666
0.3	0.273
0.4	0.010
0.5	-0.068
0.6	
0.7	
0.8	



* see computer printout for 10^{th} order filter

B find the 2-D transformation function $F(\omega_1, \omega_2) = \cos\omega$

the simplest choice of a hexagonal, circular F is the frequency response of a weighted delta plus a weighted unit hexagons



f's frequency response is:

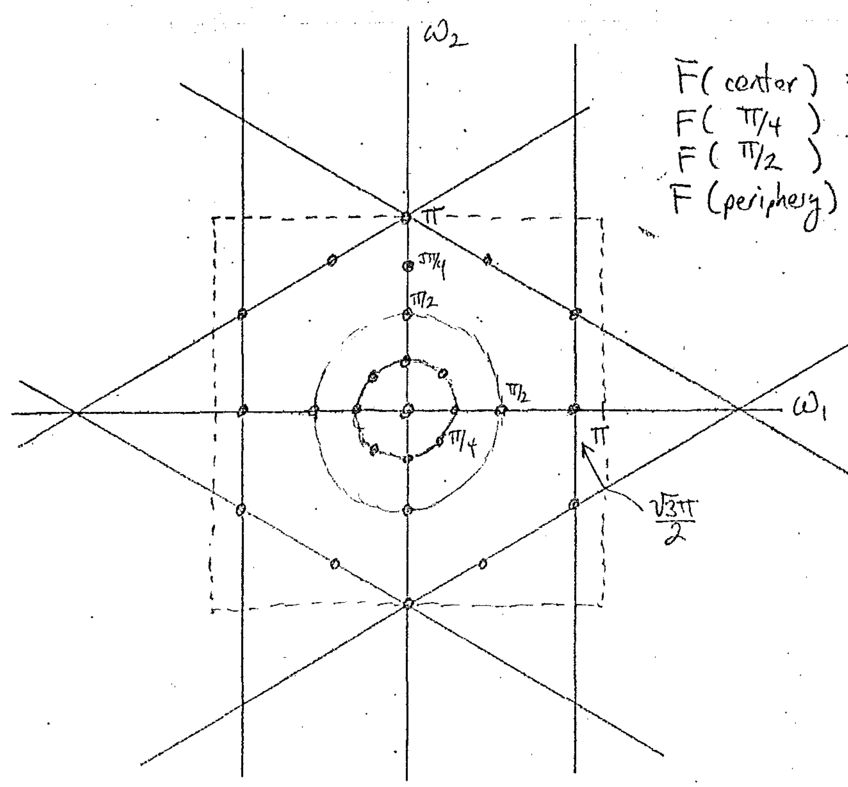
$$\begin{aligned} F(\omega_1, \omega_2) = \; & f[0,0] \\ & + f[1,0] \exp\left(-j\left(2\omega_1/\sqrt{3}\right)\right) + f[-1,0] \exp\left(-j\left(-2\omega_1/\sqrt{3}\right)\right) \\ & + f[0,1] \exp\left(-j\left(-\omega_1/\sqrt{3} + \omega_2\right)\right) + f[0,-1] \exp\left(-j\left(\omega_1/\sqrt{3} - \omega_2\right)\right) \\ & + f[1,1] \exp\left(-j\left(\omega_1/\sqrt{3} + \omega_2\right)\right) + f[-1,-1] \exp\left(-j\left(-\omega_1/\sqrt{3} - \omega_2\right)\right) \end{aligned}$$

which simplifies to:

$$F(\omega_1, \omega_2) = A + 2B\cos 2\omega_1/\sqrt{3} + 4B\cos\omega_2 \cos\omega_1/\sqrt{3}$$

an isopotential plot of this response:

$F(\text{center}) = A + 6B$
$F(\pi/4) = A + B \cdot 4.828$
$F(\pi/2) = A + 2B$
$F(\text{periphery}) = A - 2B$



B cont'd

To select values for A and B note that since

$$F(\omega_1,\omega_2) = \cos\omega, \text{ and our prototype is lowpass,}$$

F should approach 1 in the passband and -1 in the stopband.

The most natural selection of A and B therefore is

$$A = -2B$$
$$B = 1/4$$

This choice produces

$$F(\text{center, passband}) = 1$$
$$F(\text{periphery, stopband}) = -1$$

$$F(\omega_1\omega_2) = -1/2 + 1/2\cos(2\omega_1/\sqrt{3}) + \cos\omega_2\cos(\omega_1/\sqrt{3})$$

note

$F(0,\omega_2) = \cos\omega_2$ the ω_2 slice exactly corresponds

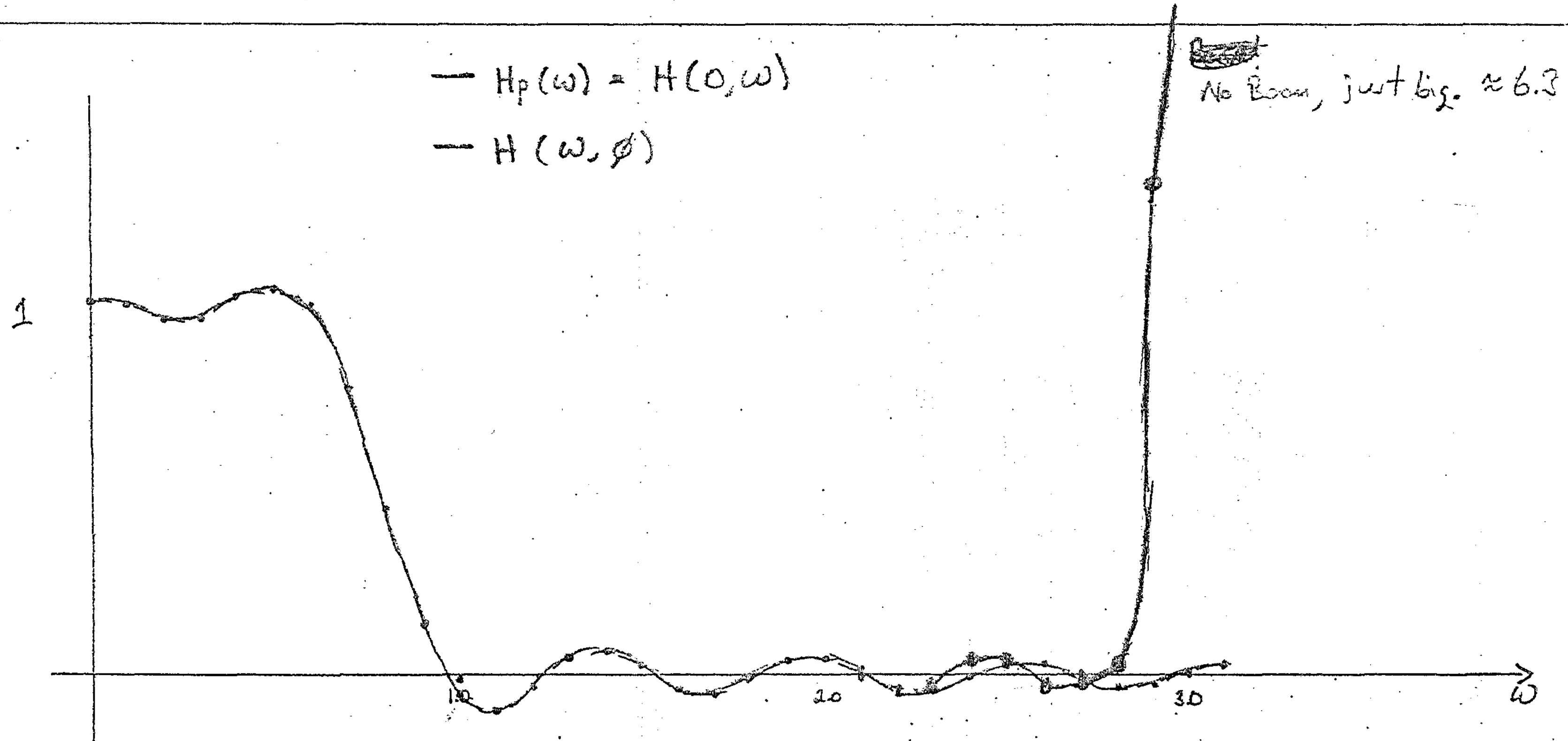
$\Rightarrow H(0,\omega_2) = H(\omega_2)$ to the prototype H

but

$$F(\omega_1,\emptyset) = -1/2 + 1/2\cos(2\omega_1/\sqrt{3}) + \cos(\omega_1/\sqrt{3})$$

so, $H(\omega_1,\emptyset)$ will be different from the prototype

C did the transform work?



notes

$H_p(\omega) = H(0, \omega)$ for all ω $-\pi < \omega < \pi$

$H_p(\omega) \approx H(\omega, 0)$ after ω exceeds 2.0 the functions diverge and H becomes relatively large perhaps modifying A and B could help some

See printout sheets. Set $A = -0.275$ $B = 0.229$ removes the "Boom" but does very bad things to the low frequency match of Hp and H

Problem 6.8:

a) $$W'(\underline{k}) = \frac{1}{N}\sum_{i=0}^{N-1} w(i) \exp\left[-j\underline{k}'(\underline{x}_i+\underline{d})\right]$$

$$= \exp[-j\underline{k}'\underline{d}] \sum_{i=0}^{N-1} w(i) \exp[-j\underline{k}'\underline{x}_i]$$

$$= W(\underline{k}) \exp[-j\underline{k}'\underline{d}]$$

where $\underline{d} \triangleq (d_x, d_y, d_z)'$

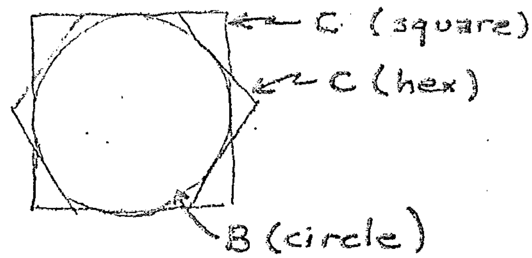
b) $$W'(\underline{k}) = \frac{1}{N}\sum_{i=0}^{N-1} w(i) \exp[-jD\underline{k}'\cdot\underline{xi}]$$

$$= W(\underline{k}D)$$

c) $$W'(\underline{k}) = \frac{1}{N}\sum_{i=0}^{N-1} w(i) \exp[-jk_xD_xx_i - jk_yD_yy_i - jk_zD_zz_i]$$

$$= W(\underline{\ell}) \text{ where } \underline{\ell} = (k_xD_x, k_yD_y, k_zD_z)'$$

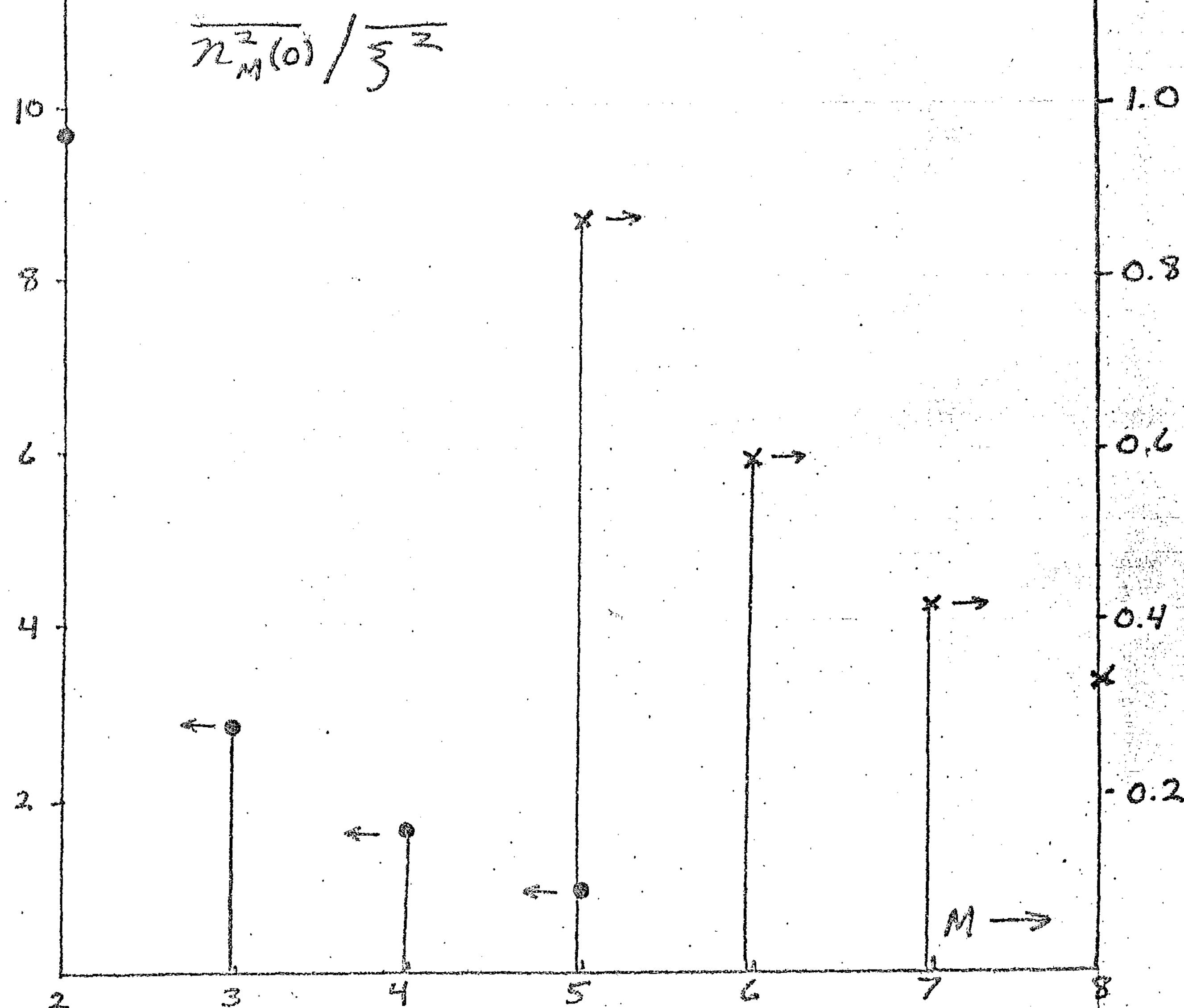
3. For (minimum density) hex sampling



$$\frac{\overline{n^2(\vec{0})}}{\overline{\xi^2}} = \left[\frac{C}{B} - 1\right]^{-1} = \left[\frac{C}{\hat{C}}\,\frac{\hat{C}}{B} - 1\right]^{-1}$$

A table of $C/\hat{C}$ is on p. 47 of text
$\hat{C} = (2p)^M$
$B =$ AREA OF M-D sphere

Note values are higher than rect. case



Problem 5.3:

a) $\dfrac{Y_z(z)}{X_z(z)} = \dfrac{1}{1-az^{-1}}$ $\qquad Y_z(z)=X_z(z)+az^{-1}Y_z(z)$

$$\boxed{c(n) = a\delta(n-1)}$$

b) $\boxed{y_i(n)=x(n)+ay_{i-1}(n-1)}$

c) $y_0(n)=\delta(n)$

$y_1(n)=\delta(n)+a\delta(n-1)$

$y_2(n)=\delta(n)+a\delta(n-1)+a^2\delta(n-2)$

$\vdots$

$$y_I(n) = \sum_{i=0}^{I} a^i\delta(n-i)$$

$\vdots$

$$y_\infty(n) = a^n u(n)$$

$$\boxed{e_2(n) = \sum_{n=I+1}^{\infty} a^{2n} = \frac{a^{2(I+1)}}{1-a^2}}$$

Lost Sample Solution

In General, for rectangular D:

$$f(\vec{t}) = \frac{|\det \underline{V}|}{(2\pi)^2} \iint_{-2\pi B}^{2\pi B} e^{j(\Omega_1 t_1 + \Omega_2 t_2)} d\Omega_1 d\Omega_2$$

$$\Omega_p = 2\pi u_p \; ; \; p = 1, 2$$

$$f(t_1, t_2) = T^2 \int_{-B}^{B} e^{j 2\pi (u_1 t_1 + u_2 t_2)} du_1 du_2$$

$$= T^2 (2B)^2 \operatorname{sinc}(2Bt_1) \operatorname{sinc}(2Bt_2)$$

For (a), $B = W$

For (b), $B = \frac{3}{2} W$ Let $B = cW$

$c = 1$ for (a)
$c = 3/2$ " (b)

$$f(t_1, t_2) = \frac{1}{16 \cancel{W^2}} \cdot 4 c^2 \cancel{W^2} \operatorname{sinc}(2cWt_1) \operatorname{sinc}(2cWt_2)$$

$$= \frac{c^2}{4} \operatorname{sinc}(2cWt_1) \operatorname{sinc}(2cWt_2)$$

$$f(\vec{0}) = \frac{c^2}{4}$$

In general:

$$x_a(0,0) = \frac{1}{1 - f(\vec{0})} \sum_{\vec{n} \neq 0} x_a(\underline{V}\vec{n}) f(-\underline{V}\vec{n})$$

For our case:

$$x_a(0,0) = \frac{c^2/4}{1 - \frac{c^2}{4}} \sum_{\vec{n} \neq 0} x_a(n_1 T, n_2 T)$$

$$\times \operatorname{sinc}\left(\frac{n_1 c}{2}\right) \operatorname{sinc}\left(\frac{n_2 c}{2}\right)$$

For given data:

$$X_a(0,0) = \frac{1}{\left(\frac{2}{c}\right)^2 - 1}\left[\sum_{n_1=-\infty}^{\infty}\sum_{n_2\,\text{odd}} \frac{2}{\pi n_2}(-1)^{\frac{n_2-1}{2}} \operatorname{sinc}^2\frac{n_1}{4}\right.$$

$$\times \operatorname{sinc}\left(\frac{n_1 c}{2}\right)\operatorname{sinc}\left(\frac{n_2 c}{2}\right)$$

$$\left. + \sum_{n_1\neq 0}\operatorname{sinc}^2\left(\frac{n_1}{4}\right)\operatorname{sinc}\left(\frac{n_1 c}{2}\right)\right]$$

$n_2 = 2m+1$

$$X_a(0,0) = \frac{1}{\left(\frac{2}{c}\right)^2 - 1}\left[\frac{2}{\pi}\left\{1 + 2\sum_{n=1}^{\infty}\operatorname{sinc}^2\left(\frac{n}{4}\right)\operatorname{sinc}\left(\frac{n_1 c}{2}\right)\right\}\right.$$

$$\times\left\{\sum_{m=-\infty}^{\infty}(-1)^m\frac{1}{(2m+1)}\operatorname{sinc}\frac{(2m+1)c}{2}\right\}$$

$$\left. + 2\sum_{p_1=1}^{\infty}\operatorname{sinc}^2\left(\frac{p}{4}\right)\operatorname{sinc}\frac{pc}{2}\right]$$

$$= \frac{2}{\left(\frac{2}{c}\right)^2 - 1}\left[\frac{1}{\pi}\left\{1 + 2\sum_{n=1}^{\infty}\operatorname{sinc}^2\left(\frac{n}{4}\right)\operatorname{sinc}\left(\frac{nc}{2}\right)\right\}\right.$$

$$\times\left\{\operatorname{sinc}\frac{c}{2} + \sum_{m=1}^{\infty}(-1)^m\left[\frac{\operatorname{sinc}\frac{(2m+1)c}{2}}{2m+1} - \frac{\operatorname{sinc}\frac{(2m-1)c}{2}}{2m-1}\right.\right.$$

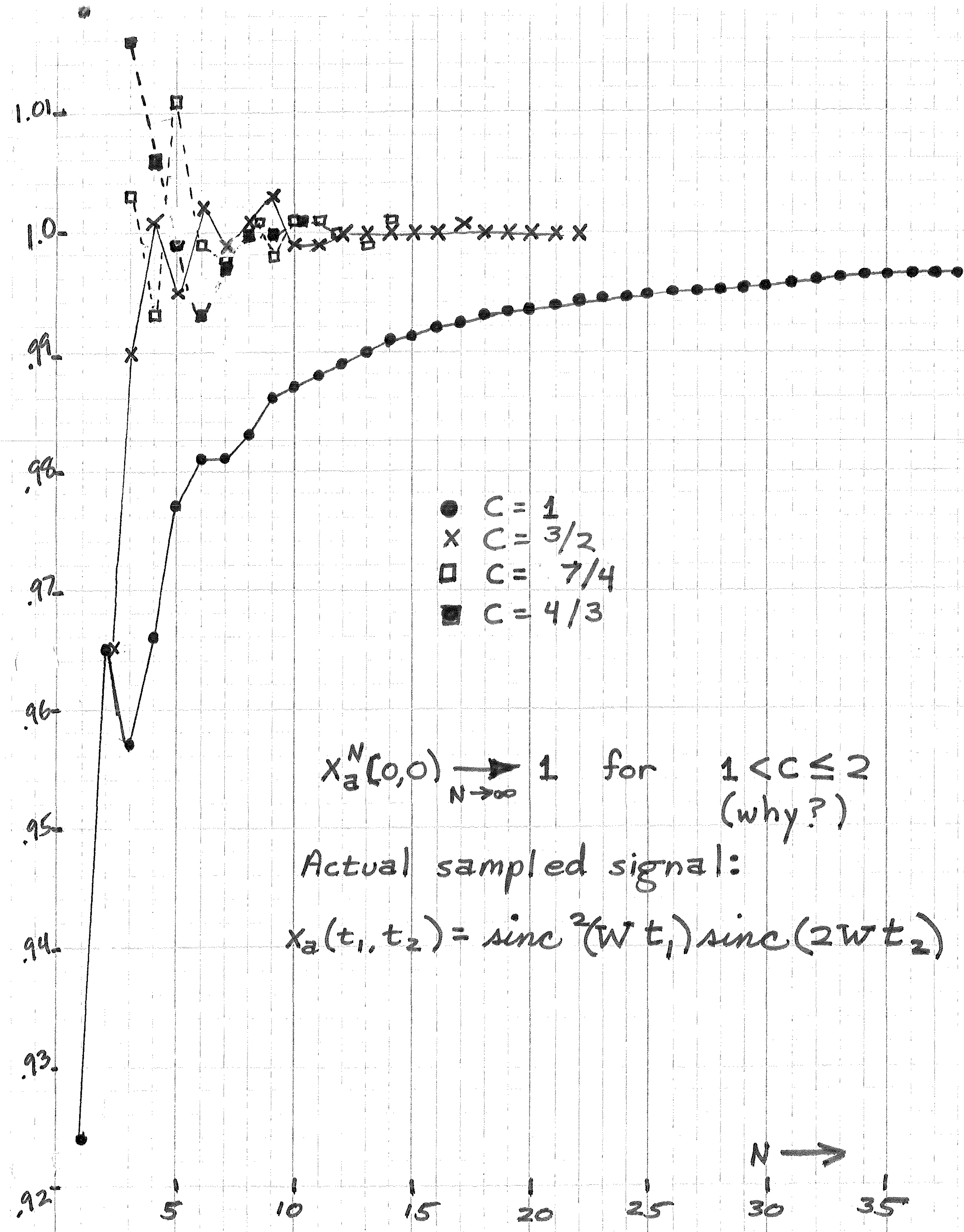
$$\left. + \sum_{p=1}^{\infty}\operatorname{sinc}^2\left(\frac{p}{4}\right)\operatorname{sinc}\left(\frac{pc}{2}\right)\right]$$

Define:

$$X_a^N(0,0) = \frac{2}{\left(\frac{2}{c}\right)^2 - 1}\left[\frac{1}{\pi}\left\{1 + 2\sum_{n=1}^{N}\operatorname{sinc}^2\left(\frac{n}{4}\right)\operatorname{sinc}\left(\frac{nc}{2}\right)\right\}\right.$$

$$\times\left\{\operatorname{sinc}\frac{c}{2} + \sum_{m=1}^{N}(-1)^m\left[\frac{\operatorname{sinc}\frac{(2m+1)c}{2}}{2m+1} - \frac{\operatorname{sinc}\frac{(2m-1)c}{2}}{2m-1}\right]\right\}$$

$$\left. + \sum_{n=1}^{N}\operatorname{sinc}^2\left(\frac{n}{4}\right)\operatorname{sinc}\left(\frac{nc}{4}\right)\right]$$


1.01
1.0
.99
.98
.97
.96
.95
.94
.93
.92
5
10
15
20
25
30
35
N →
● C = 1
× C = 3/2
□ C = 7/4
■ C = 4/3
$x_a^N(0,0) \xrightarrow[N\to\infty]{} 1$ for $1 < C \leq 2$
(why?)
Actual sampled signal:
$x_a(t_1, t_2) = \mathrm{sinc}^2(W t_1)\,\mathrm{sinc}(2W t_2)$

Noise sensitivity of the sampling theorem:

$$x_a(\vec{t}) = \sum_{\vec{n}} x_a(\underline{V}\vec{n}) f(\vec{t} - \underline{V}\vec{n})$$

Let

$$\varphi(\vec{t}) = \sum_{\vec{n}} \cancel{x_a} \xi(\underline{V}\vec{n}) f(\vec{t} - \underline{V}\vec{n})$$

Then

$$\overline{\varphi^2(\vec{t})} = \sum_{\vec{n}} \sum_{\vec{m}} R_{\xi}\left(\underline{V}(\vec{n} - \vec{m})\right) f(\vec{t} - \underline{V}\vec{n}) f(\vec{t} - \underline{V}\vec{m})$$

Assume sample wise white:

$$R_{\xi}(\underline{V}(\vec{n} - \vec{m})) = \overline{\xi^2} \; \delta(\vec{n} - \vec{m})$$

Then

$$\overline{\varphi^2(t)} = \overline{\xi^2} \sum_{\vec{m}} f(\vec{t} - \underline{V}\vec{m}) f(\vec{t} - \underline{V}\vec{m})$$

Clearly, from the original sampling theorem

$$f(\vec{\tau} - \vec{t}) = \cancel{\overline{\xi^2}} \sum_{\vec{m}} f(\vec{\tau} - \underline{V}\vec{m}) f(\vec{t} - \underline{V}\vec{m})$$

Thus, for $\vec{\tau} = \vec{t}$

$$\varphi^2(\vec{t}) = \overline{\xi^2} \; f(\vec{0}) \quad \bigstar$$

~~Can we reduce this? Since~~

$$\cancel{x_a(\vec{t}) = x_a(\vec{t}) * f(\vec{t})}$$

~~we define~~

$$\cancel{\psi(\vec{t}) = \varphi(\vec{t}) * f(\vec{t})}$$

But

$$f_A(x) = 2\int_{-\infty}^{\infty} \frac{r\, f(r)\, \mu(r-x)\, dr}{\sqrt{r^2 - x^2}}$$

set $\xi = x^2$

$$\rho = r^2 \Rightarrow r = \sqrt{\rho} \Rightarrow dr = \frac{d\rho}{2\sqrt{\rho}}$$

$$f_A(\sqrt{\xi}) = \cancel{2}\int_{-\infty}^{\infty} \frac{\cancel{\sqrt{\rho}}\, f(\sqrt{\rho})\, \mu(\sqrt{\rho} - \sqrt{\xi})\, \frac{d\rho}{\cancel{2\sqrt{\rho}}}}{\sqrt{\rho - \xi}}$$

$$\mu(\sqrt{\rho} - \sqrt{x}) = \mu(\rho - \xi)$$

$$f_A(\sqrt{\xi}) = \int_{-\infty}^{\infty} f(\sqrt{\rho})\, \frac{\mu(\rho - \xi)}{\sqrt{\rho - \xi}}\, d\rho$$

$$= f(\sqrt{\xi}) * \frac{\mu(-\xi)}{\sqrt{-\xi}}$$

Fourier Theorem:

$$\int_{-\infty}^{\infty} a(\rho)\, b(\xi - \rho)\, d\rho \longleftrightarrow A(\omega)\, B(\omega)$$

$$A(\omega) = \int_{-\infty}^{\infty} a(\xi)\, e^{-j\xi\omega}\, d\xi$$

$$\mathcal{F}[f_A(\sqrt{\xi})] = \mathcal{F}[f(\sqrt{\xi})]\; \mathcal{F}\left[\frac{\mu(-\xi)}{\sqrt{-\xi}}\right]$$

$$= \mathcal{F}[f(\sqrt{\xi})] \cdot \sqrt{\frac{j\pi}{\omega}}$$

Thus:

$$\mathcal{F}[f(\sqrt{\xi})] = \mathcal{F}[f_A(\sqrt{\xi})] \cdot \sqrt{\frac{\omega}{j\pi}}$$

But: $$\sqrt{\frac{\omega}{j\pi}} = j\omega \sqrt{\frac{j\pi}{\omega}} \cdot \left(-\frac{1}{\pi}\right)$$

$$= \cancel{\sqrt{\frac{-j\omega}{\pi}}}$$

$$= \cancel{j\frac{1}{\pi}\sqrt{j\pi}\,\omega}$$

EE521 name______________

examination #1

closed book, no scratch paper (there's some at the end of the booklet).
one sheet of notes, a calculator and a math table are okay.
please do all of your work in the test booklet.

all problems are worth 25 points.

****

PROBLEM 1:

Consider the following periodicity matrix:

$$V = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

Which of the following matrices produce the same periodic replication? Choose all that apply.

(a) $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & -1 \\ -1 & -1 \end{bmatrix}$

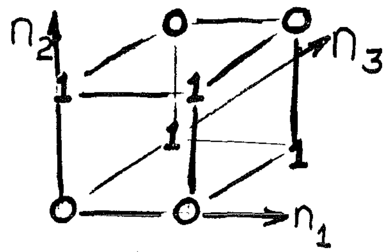
(d) $\begin{bmatrix} 87 & 1 \\ 1 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$ (f) $\begin{bmatrix} 2 & -1 \\ 2 & 1/2 \end{bmatrix}$

Circle the equivalent matrices clearly. Ambiguous answers will be graded as incorrect.

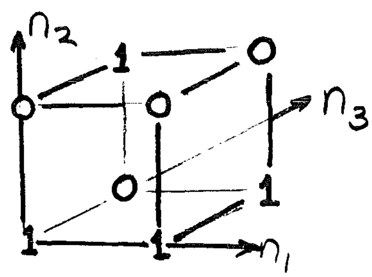
PROBLEM 2:

Consider the two three dimensional signals shown below. The value of both functions at all points is either one or zero. The value of the function is shown at its location. If a value is not shown, it is zero. In both cases, the origin is the lower left front corner of the cube. Let $y = x * h$. Compute $y(1,1,0)$.

$x(n_1,n_2,n_3)$



$h(n_1,n_2,n_3)$



PROBLEM 5:

The half order derivative of a function is obtained by multiplying the spectrum of a signal by the square root of j omega and inverse transforming. Using this insight , derive the function that, when convolved with x(t), will result in its half derivative.

Official Scratch Paper:

Official Scratch Paper:

Official Scratch Paper:

LE 521

When N points of subdivision are used, the scale of ρ is arranged so that F becomes zero at $\rho = N$. The coefficients may then all be multiplied by $(10/N)^{\frac{1}{2}}$, or the coefficients may be left unchanged and the answers multiplied by $(10/N)^{\frac{1}{2}}$.

As an example consider $F(\rho) = (10 - \rho)^{\frac{1}{2}}$, for which the modified Abel transform is known to be $F_A(\xi) = \frac{1}{2}\pi(10 - \xi)$. We work at unit intervals and copy the coefficients on a movable strip. The calculation in progress is shown in Fig. 12.7. The movable strip is in position for calculating $F_A(\xi)$ as the sum of products of corresponding values of F and K:

$$7.78 = 2.12 \times 2.000 + 1.87 \times 0.828 + \ . \ . \ . \ + 0.71 \times 0.472.$$

The inverse problem, that of calculating F from F_A, can be handled by means of the relation $F = -\pi^{-1}K * F'_A$ if F_A is first differentiated. However, it will be perceived that the calculation just described can be done in reverse, using the values of F_A, and working the movable strip upward from the bottom. The strip is shown in position for calculating $F(5 - \frac{1}{2})$, let us say by means of a pocket calculator. Form the products $0.71 \times 0.472, \ldots, 1.87 \times 0.828$, allowing them to accumulate in the memory. Subtract this sum of products from 7.78 and divide by 2.000 to obtain the next wanted value, $F(5 - \frac{1}{2}) = 2.12$. The inverse transformation can be performed quickly in this way.

ρ	F	K	F_A
			15.65
	3.08		
1			14.08
	2.91		
2			12.52
	2.74		
3			10.94
	2.55		
4			9.37
	2.35		
5			7.78
	2.12	2.000	
6			6.20
	1.87	0.828	
7			4.62
	1.58	0.636	
8			3.03
	1.22	0.536	
9			1.42
	0.71	0.472	
10			0
		0.427	

Fig. 12.7 Calculating modified Abel transforms.

EE521 name *Solutions*
Friday, December 9, 1994
10:30 AM

INSTRUCTIONS:
* Do all of your work in this test booklet.
* This test is closed book and closed note.
* You are allowed two legal sized sheets of notes & a calculator.
* Each problem is worth 25 points.

//

1. A half order derivative, $(d/dt)^{1/2}\, x(t)$, can be written in integral form as

$$(d/dt)^{1/2}\, x(t) = \int x(\tau)\, k(t;\tau)\, d\tau$$

where integration is over all τ. Evaluate the kernel, $k(t;\tau)$.

$$\left(\frac{d}{dt}\right)^{\frac{1}{2}} x(t) \longleftrightarrow (j\omega)^{\frac{1}{2}} X(\omega)$$

Recall: $\dfrac{\mu(-t)}{\sqrt{-t}} \longleftrightarrow \sqrt{\dfrac{j\pi}{\omega}}$

Thus: $(j\omega)^{\frac{1}{2}} X(\omega) = \dfrac{1}{j\sqrt{\cancel{j}\pi}}\, j\omega \times \sqrt{\dfrac{j\pi}{\omega}}\, X(\omega)$

The inverse transform is

$$\cancel{\left(\frac{d}{dt}\right)^{\frac{1}{2}} X(\omega)}$$

$$\left(\frac{d}{dt}\right)^{1/2} x(t) = \frac{1}{j\sqrt{\pi}} \frac{d}{dt}\left(\frac{\mu(-t)}{\sqrt{-t}}\right) * x(t)$$

$$= \frac{1}{\sqrt{\pi}} \frac{d}{dt} \frac{\mu(-t)}{\sqrt{t}} * x(t)$$

Since $\frac{d}{dt}\mu(-t) = -\delta(t)$

$$\left(\frac{d}{dt}\right)^{\frac{1}{2}} x(t) = \frac{-1}{\sqrt{\pi}}\left[\frac{\delta(t)}{\sqrt{t}} + \frac{\mu(-t)}{2t^{3/2}}\right] * x(t)$$

~~and~~

$$= \frac{-1}{\sqrt{\pi}} \int_{-\infty}^{\infty} x(\tau)\left[\frac{\delta(t-\tau)}{\sqrt{t-\tau}} + \frac{\mu(\tau-t)}{2(t-\tau)^{3/2}}\right] d\tau$$

and:

$$\cancel{k(\tau)=}$$

$$k(t;\tau) = \frac{-1}{\sqrt{\pi}}\left[\frac{\delta(t-\tau)}{\sqrt{t-\tau}} + \frac{\mu(\tau-t)}{2\sqrt{(t-\tau)^3}}\right]$$

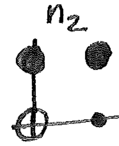
$$= \frac{-1}{\sqrt{\pi}\sqrt{t-\tau}}\left[\delta(t-\tau) + \frac{\mu(\tau-t)}{2(t-\tau)}\right]$$

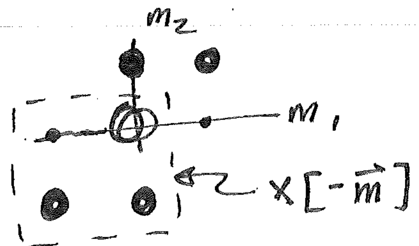
a. Evaluate the circular convolution of the following 2-D signal with itself:

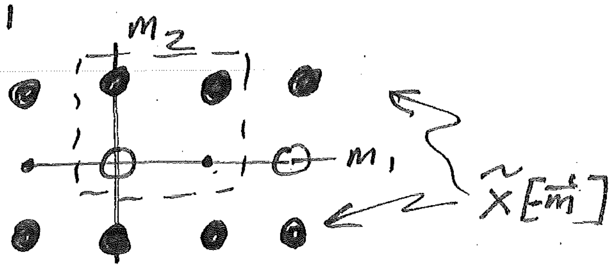
$x[0,0] = -1$
$x[1,0] = 0$
$x[0,1] = 1$
$x[1,1] = 1$

b. Can a circular convolution of a function, other than one identically zero, with itself result in a function that is identically zero? If so, give an example.

(a)



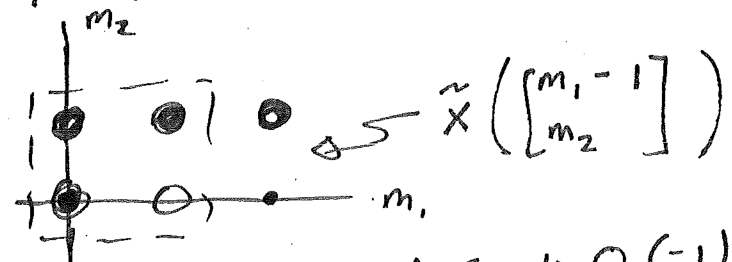




$y[n_1, n_2] = x \circledast x$

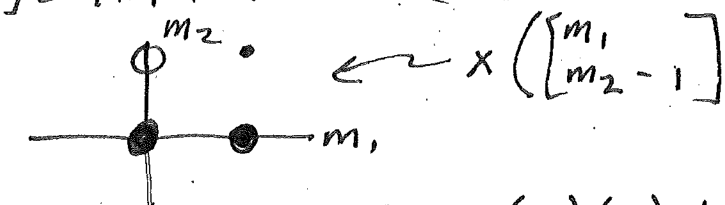
① $y[0,0] = 1 \cdot 1 + 1 \cdot 1 + (-1)(-1) + 0 \times 0 = 3$

② $y[1,0]$



$y[1,0] = 1 \times 1 + 1 \times 1 + (-1) \cdot 0 + 0(-1) = 2$

③ $y[0,1]$

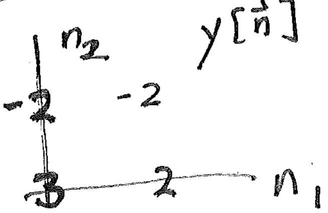


$y[0,1] = (1)(-1) + (1)(0) + (-1)(1) + (0)(1) = -2$

④ $y[1,1]$

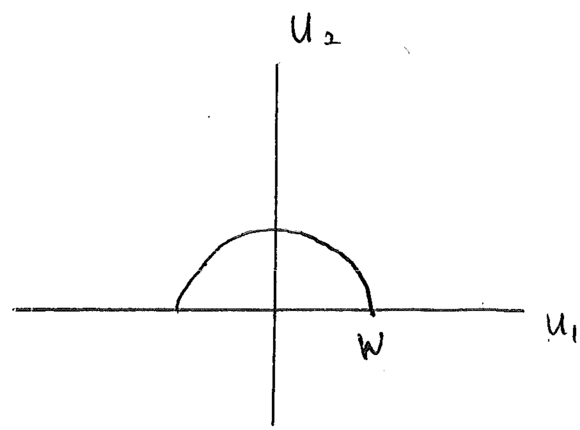
$y[1,1] = (1) \cdot (0) + (1)(-1) + (-1)(1) + (0 \cdot 1)$
$= -2$

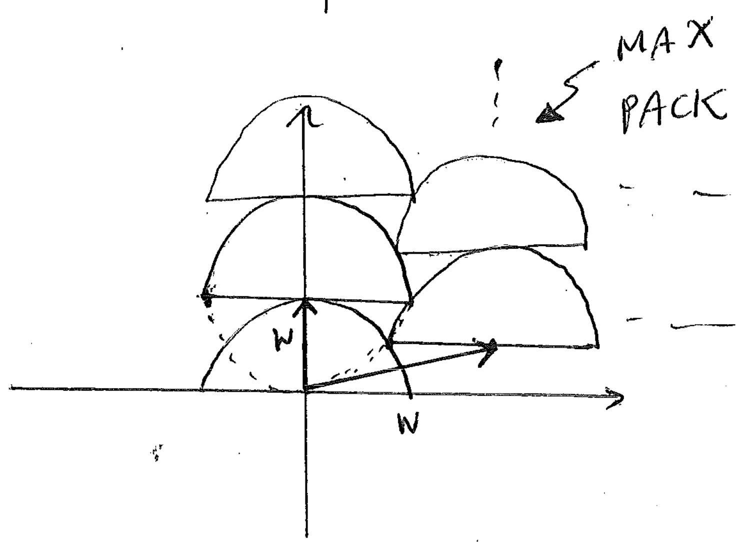
Solution



(b) No Can only be true if DFT (or FFT) of $x[\vec{n}]$ is zero. This is true only when $\tilde{X}[\vec{n}] \equiv 0$

4. A two dimensional signal has a Fourier transform that is identically zero outside of a <u>half</u> circle with radius W. Evaluate the corresponding Nyquist density.





$$\mathbb{P} = \begin{bmatrix} 0 & (1+\frac{\sqrt{3}}{2})W \\ W & \frac{1}{2}W \end{bmatrix}$$

$$\Rightarrow \quad SD_{NYQ} = \left(1+\frac{\sqrt{3}}{2}\right)W^2 \,!$$

(a) Consider the operation of transposing a function. That is, from $x(\mathbf{t})$, we make $x(-\mathbf{t})$ where $\mathbf{t}$ is a vector. Is this operation linear? Is is shift invariant? Explain your reasoning in each case.

(b). Give an example of a system that is additive but not homogeneous.

(a) Linear $\Rightarrow$ yes

$$y(t) = S\,x[t] = x(-t)$$

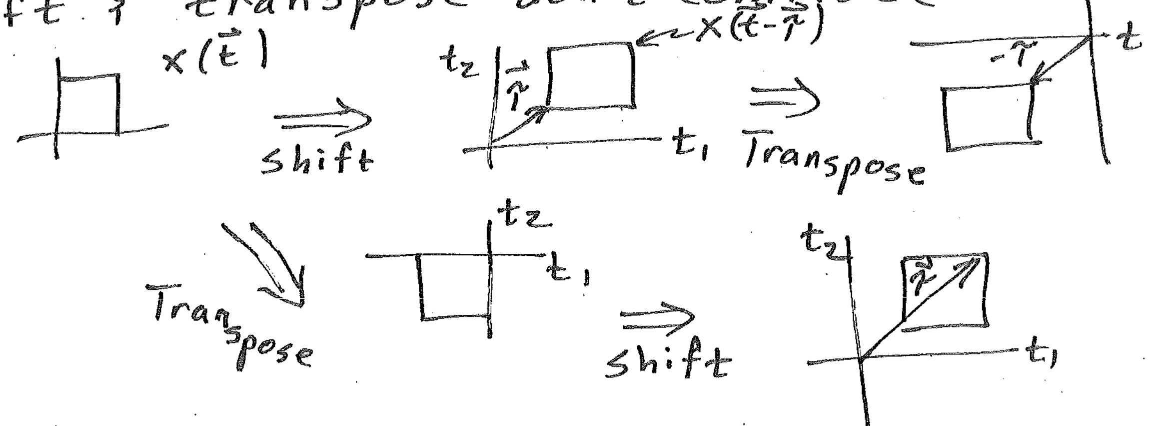
Additivity: $S\,a\,x(t) = a\,x(-t)$

Homo: $S\,x_1(t) + x_2(t) = x_1(-t) + x_2(-t)$

$\therefore$ Linear

<u>Not</u> shift invariant

Shift & transpose don't commute



(b) How about

$$y = S\,x = \cancel{\ } x^*$$

① additive: $S\,x_1 + x_2 = x_1^* + x_2^* = S\,x_1 + S\,x_2$

② Homo: $S\,a\,x = a^*\,x^* \neq a\,S\,x = a\,x^*$

A three dimensional signal, $x(n_1,n_2,n_3)$, is zero everywhere except the first octant (where all three variables are not negative). In the first octant, the function is

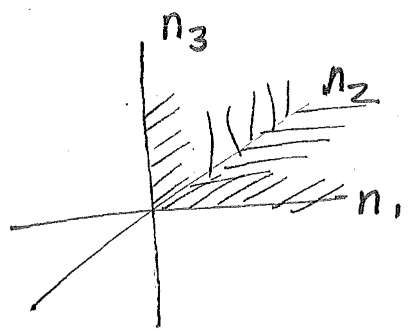
$$x(n_1,n_2,n_3) = \quad (1/2)\wedge(n_1+n_2+n_3)$$

If $x(n_1,n_2,n_3) = h(n_1,n_2,n_3)$ and

$$y(n_1,n_2,n_3) = x(n_1,n_2,n_3) * h(n_1,n_2,n_3),$$

where * denotes convolution, what is $y(0,0,0)$?

HINT: CONSIDER THE CONVOLUTION MECHANICS.



Transpose in 3-D.
Only one non-zero point overlapping (origin)

$x(0,0,0) = 1$

$\Rightarrow y(0,0,0) = 1$

Elementary Finance Analysis Using Difference Equations and z-Transforms

Robert J. Marks II

1 Introduction

Many common problems involving interest in personal finance can be solved by

1. writting, by inspection, a describing difference equation, and
2. solving the difference equation using a unilateral z-transform.

Examples given in this monograph include analysis of

- compound interest on a simple deposit,
- compound interest on periodic deposits, and
- payment scheduling of loans, such as morgages, where premiums are paid periodically, and
- effects of taxes and inflation.

1.1 Some Preliminary Math

1.1.1 Unilateral z-Transforms

The *unilateral* z-transform of a sequence $x[n]$ is [1]

$$X(z) = \sum_{n=0}^{\infty} x[n]\, z^{-n}$$

The transform pair can be written in short hand as

$$x[n] \leftrightarrow X(z)$$

For example

$$a^n \mu[n] \leftrightarrow \frac{1}{1 - az^{-1}} \tag{1}$$

[1] When the summation over n is over the interval $(-\infty, \infty)$, the z transform is said to be *bilateral*.

2 Compound Interest on a One Time Deposit.

Interest quotes have two components.

- annual interest and
- the frequency of compounding.

Let r be the annual interest and N the number of times per year compounding occurs. If $N = 12$, as is the case with most passbook savings, compounding is performed monthly.

A one time deposite of d is made in an account that yields an interest of r compounded N times per year. Let $\hat{b}[n]$ be the balance at the end of the n^{th} period. The difference equation describing the accumulating interest is

$$\hat{b}[n+1] = \left(1 + \frac{r}{N}\right)\hat{b}[n] \tag{6}$$

with the initial condition $\hat{b}[0] = d$. This is a special case of the difference equation in Equation 3 with

$$\begin{aligned} x[n] &\rightarrow \hat{b}[n] \\ \xi &\rightarrow 1 + \frac{r}{N} \\ \eta &\rightarrow 0 \\ x_0 &\rightarrow d \end{aligned}$$

Making these substitutions in Equation 4 gives the balance at the end of the n^{th} compounding period as

$$\hat{b}[n] = d\left(1 + \frac{r}{N}\right)^n.$$

The balance at the end of a year is

$$\hat{b}[N] = d\left(1 + \frac{r}{N}\right)^N \tag{7}$$

and at the end of M years is

$$\hat{b}[NM] = d\left(1 + \frac{r}{N}\right)^{NM} \tag{8}$$

This is a "zero over zero" situation to which we can apply L'Hopital's rule. [3]

$$\lim_{N\rightarrow\infty} \ln\left(1 + \frac{r}{N}\right)^N = \lim_{N\rightarrow\infty} \frac{\frac{d}{dN}\ln\left(1 + \frac{r}{N}\right)}{\frac{d}{dN}\left(\frac{1}{N}\right)} = r.$$

This completes the proof.

Thus

$$1 + r \le \frac{\hat{b}[N]}{d} \le e^{r}.$$

Note that for modest interest rates, the spread is very small since, for $r \ll 1$,

$$e^{r} \approx 1 + r. \tag{15}$$

2.5 Effect of annual taxes.

Consider the same problem of evaluating the balance of a one time deposit of d, except that the interest each year is taxed at a rate, t. Let $f[M]$ be the balance after year M before taxation and $c[M]$ be the balance after year M after taxation. The before taxation balance at year $M + 1$ is given by Equation 7 with $d \rightarrow c[M]$.

$$f[M+1] = c[M]\left(1 + \frac{r}{N}\right)^{N}.$$

The taxable interest earned in year M is new balance minus the initial balance.

$$i[M] = f[M+1] - c[M]$$

The amount payed in taxes is $t \times i[M]$. The after tax balance is

$$c[M+1] = f[M+1] - t \times c[M]$$

Substituting the previous two equations results in the difference equation

$$c[M+1] = \left[(1-t)\left(1 + \frac{r}{N}\right)^{N} + t\right] c[M].$$

This is a special case of the difference equation in Equation 3 with

$$\begin{aligned} n &\rightarrow M \\ x[n] &\rightarrow c[M] \\ \xi &\rightarrow (1-t)\left(1 + \frac{r}{N}\right)^{N} + t \\ \eta &\rightarrow 0 \\ x_0 &\rightarrow c[0] = d \end{aligned}$$

Making these substitutions in Equation 4 gives the desired result.

$$c[M] = d\left[(1-t)\left(1 + \frac{r}{N}\right)^{N} + t\right]^{M} \tag{16}$$

2.5.1 Continuous Compounding.

Imposing the limit in Equation 5 onto Equation 16 gives the continuous compounding solution

$$\lim_{N \rightarrow \infty} c[M] = d\left[(1-t)e^{r} + t\right]^{M} \tag{17}$$

2.5.2 Extrema.

As a function of N, Equation 16 is minimum for $N = 1$ and maximum for $N = \infty$. Thus, from Equation 17, the following extrema of yield results.

$$[(1-t)(1+r)+t]^M \leq \frac{c[M]}{d} \leq [(1-t)e^r + t]^M$$

From Equation 15, for modest interest rates ($r \ll 1$) and moderate M, these bounds are tight.

2.5.3 Combining the tax and interest rates into an equivalent interest rate.

For a given tax rate, t, and compounding frequency, N, an equivalent (smaller) interest rate, r_t, exists. Equating Equations 16 and 8 gives

$$\left[(1-t)\left(1+\frac{r}{N}\right)\right]^M d = \left(1+\frac{r_t}{N}\right)^{NM} d. \tag{18}$$

Solving for r_t gives

$$r_t = (1-t)\left[\left(1+\frac{r}{N}\right)^N - 1\right]. \tag{19}$$

The equivalent instantaneous compounding interest rate from a taxed instantaneous interest rate follows from application of Equation 5 to Equation 19.

$$\lim_{N \to \infty} r_t = (1-t)\left(e^r - 1\right)$$

2.6 Effect of inflation.

A constant inflation rate can be viewed as a negative interest rate. If u is the rate of inflation, the effect of inflation on d dollars over one year is given by Equation 10 making the replacement $r \to -u$.

$$de^{-u}$$

Over M years, the balance has reduced to

$$\left[de^{-u}\right]^M = de^{-Mu}.$$

For example, if you stuffed $d = \$100$ in your matress for $M = 3$ years, its purchasing value, at an annual inflation rate of 12%, is diminished to

$$\$100 \times e^{-3 \times 0.12} = \$69.77$$

in terms of the purchasing value of money at the time of the initial deposit.

Adjustment for inflation can be assessed after yield is evaluated. Two examples follow.

the end of n periods[5], the describing difference equation is

$$\hat{b}[n+1] = \left(1 + \frac{r}{n}\right)\hat{b}[n] + s. \tag{22}$$

Assume the account starts with a balance of $\hat{b}[0] = 0$. Equation 22 is then a special special case of Equation 3 with

$$\begin{aligned} x[n] &\rightarrow \hat{b}[n] \\ \xi &\rightarrow 1 + \frac{r}{N} \\ \eta &\rightarrow s \\ x_0 &\rightarrow 0 \end{aligned} \tag{23}$$

Substituting thes parameters into Equation 4 gives

$$\hat{b}[n] = \frac{Ns}{r}\left\{\left(1 + \frac{r}{N}\right)^n - 1\right\}.$$

The balance after one year is thus

$$\hat{b}[N] = \frac{Ns}{r}\left\{\left(1 + \frac{r}{N}\right)^N - 1\right\}. \tag{24}$$

and the balance after M years is

$$\hat{b}[MN] = \frac{Ns}{r}\left\{\left(1 + \frac{r}{N}\right)^{MN} - 1\right\}. \tag{25}$$

3.1 Continuous time solution.

For the continuous time solution to this problem, assume y is invested yearly in equal installments. Thus

$$s = \frac{y}{N}$$

For M years, the balance in Equation 25 therefore becomes

$$\hat{b}[MN] = \frac{y}{r}\left\{\left(1 + \frac{r}{N}\right)^{MN} - 1\right\}.$$

Using Equation 5, the balance using continuous time compounding is

$$\lim_{N\to\infty} \hat{b}[MN] = \frac{y}{r}\left(\mathrm{e}^{rM} - 1\right).$$

[5] The notation $\hat{b}$ will be used for the case of constant periodic deposits as opposed to $b[n]$ which denotes the accumulated balance on a single deposit.